ANALYTIC GEOMETRY

by

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Preface

The aim of this text is to give such a presentation of the materials of Analytic Geometry that a student will be led to a deep understanding of underlying principles while he is learning to apply them in solving the ordinary problems of the subject.

A general principle or definition has content for a student in proportion to the breadth of his comprehension of illustrations and applications of it. If, from time to time, a student has solved a number of problems such as finding the intersections, (1,2) and (2,1), of $x + y - 3 = 0$, $x^2 + y^2 - 5 = 0$ and showing that they satisfy

$$x + y - 3 + k(x^2 + y^2 - 5) = 0,$$

he will easily understand the argument used to prove that the graph of

$$u(x,y) + k\,v(x,y) = 0$$

contains the intersections of $u(x,y) = 0$ and $v(x,y) = 0$. On the other hand, he may memorize the definition of a graph, and fail to see the proof of the statement just made, because the definition is mere sound to him and not a working principle that he has used again and again. The authors of this text have attempted to give a background for thorough understanding of the general definitions and principles of the subject. The remaining part of this preface will suggest the means of accomplishing this result.

A basic principle in analytic geometry is that

$$AB + BC = AC,$$

provided that $A,B,$ and C are points on a straight line and directions are considered. This is a vector equation, and the student will not understand it thoroughly unless he sees it as such. To this end vectors have been introduced at the start. After studying the very simple law of adding vectors, the student understands the equation above as a special case of vector addition and is prepared to understand the whole concept of positive and negative numbers and their associations with directions along lines—

that is, the vector background of analytic geometry. Besides, vectors deserve emphasis because of their extreme importance in physical applications.

Each highly important and useful technique is introduced at the first opportunity, is emphasized, and then is brought in again and again when it may be applied to simplify a procedure. Consider, for example, the translation of axes. This subject is introduced in the second chapter to simplify graphing and to discover symmetry with respect to lines parallel to the axes of coordinates. Later it is used in finding the distance from a point to a line, in simplifying the equations of circles, in conic sections, in trigonometric graphs, and frequently in solid analytic geometry. Translation of axes is introduced so many times that it will become a working tool which the student will use with ease and sureness; it will exist in his mind properly related to the whole subject. This plan of early introduction and repeated use from time to time is vital in the process of assimilation of knowledge.

An attempt has been made to give each subject proper emphasis. Consider polar coordinates as an example. In most texts, this topic is relegated to an unimportant position. In this text it is introduced before conic sections; later it is considered with each type of conic section; it is brought up again under the treatment of special curves, with an important extension on asymptotes; finally it appears in connection with the cylindrical coordinates of solid analytic geometry. Here again the repeated recall, use, and amplification take advantage of the natural growth of ideas in the mind and fix these ideas in their vital relations to the other parts of the subject.

Special effort has been put forth to obtain simplicity, both from the logical and the pedagogical standpoint. For example, in solid analytic geometry as well as in plane, graphing precedes the more difficult ideas because it is simple and gives the student a familiarity with the use of coordinates which prepares the way for understanding more abstruse ideas. Also, numerous figures give the maximum value of visual aids by showing important relations at a glance and motivating the various topics.

The problem lists are worthy of mention. Many easily and quickly answered questions call attention to important features; the problems of average difficulty, and the difficult problems,

marked by stars (★), furnish abundant material to be used in acquiring skill and deeper understanding.

It is a pleasure to thank our colleagues at the Naval Academy for the give-and-take discussions which have helped in shaping our ideas. We are especially indebted to Professor George A. Lyle for his generosity in giving excellent suggestions.

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CHAPTER I

Basic Concepts

1. Introduction

It is interesting to see how the components of analytic geometry, algebra and geometry, were developed more or less separately and then combined in one subject. The Greeks developed elementary geometry and perfected it as far as the limitations of their methods would permit. One is likely to ask why, with their extremely fertile minds and their keen appreciation of the exalted position of number, they failed to develop algebra. The answer generally given is that they adopted an unlucky system of numeration in which all letters of the alphabet were used in denoting particular numbers. Accordingly, they had a poor chance of inventing an algebra of symbols in which a letter may represent any number, and they did not develop a good arithmetic. Their lack of algebra and a basic arithmetic effectively cut off their chance of discovering modern mathematical theories.

The Hindus developed a considerable amount of algebraic knowledge and perhaps the system of numeration of modern times. The Arabs, keenly appreciating the work of others, obtained the mathematical knowledge of the Greeks and Hindus. This knowledge was brought into Western Europe mainly by way of Spain during the course of the twelfth century. By 1200 Christendom had books dealing with the algebra and arithmetic of the Hindus and Arabs, and also Euclid's *Elements*, that is, the elementary geometry of the Greeks. By 1450 this knowledge was assimilated and the newly invented printing press was available for disseminating it. By 1600 the fundamental principles of arithmetic and of algebra with its symbols were laid down and the Greek geometry was widely known. The time was then ripe for a combination of the two subjects, algebra and geometry. In 1637 René Descartes, a renowned mathematician and philosopher,

invented analytic geometry and may therefore be considered the first mathematician of the modern school.* By means of analytic geometry a curve (geometric) is defined by an equation (algebraic) in two unknowns, x and y. Also, if a defining property of a plane curve is stated, this property can generally be expressed in the form of an equation, and from this equation the curve can be plotted. The equation can be used to discover properties of the curve, and often the curve can be used to find useful information inherent in the equation. Similar remarks apply to equations involving three variables and to curves and surfaces in space.

The light thrown on geometry by the methods of analytic geometry is very searching. Not only do the equations furnish a method of solving the difficult problems of geometry by a rather simple routine procedure, but, what is far more important, they are the basis of a method of discovering the properties of any kind of curve. Also, the analytical approach is very suggestive; so much so, in fact, that analytic geometry is used almost everywhere in modern mathematics and its applications. In a sense this invention of Descartes and Fermat pointed the way to the discovery of Calculus, a subject containing the essentials of most of the great modern developments of mathematics. From the start it practically superseded the geometry of the ancients so far as research is concerned, and it has maintained its eminent position as a necessary part of the training of every mathematician since its invention.

This chapter will be devoted to basic ideas and concepts of the subject.

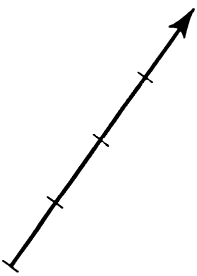

FIG. 1.

2. Vectors

Direction and length are fundamental in Analytic Geometry. The concept of a directed line segment, or vector, involves both in a useful form.

A **vector** (*see Figure 1*) *is a line segment with an arrowhead or barb on it; the length of the line segment indicates magnitude and the barbed line indicates direction.* Here the word *direction* has reference, not

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* Pierre de Fermat, a great French mathematician, invented Analytic Geometry independently of Descartes and approximately at the same time.

only to parallelism with a line, but also to the sense along the line of the vector.

We shall denote a directed line segment, or vector, from A to B by $\overrightarrow{AB}$, call it the vector $\overrightarrow{AB}$, and denote its magnitude by $|\overrightarrow{AB}|$. It may be thought of as generated by a point moving from the first-named point A along the segment AB to the second-named point B.

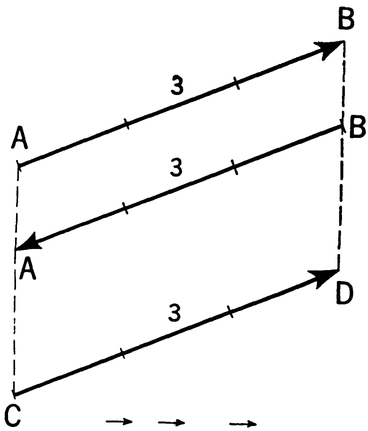


FIG. 2.

The magnitudes of the vectors in Figure 2 are all 3, and we write $|\overrightarrow{AB}| = |\overrightarrow{BA}| = |\overrightarrow{CD}| = 3$. Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ (see Figure 2) are said to be equal if they have the same magnitude and the same direction. If two vectors have the same magnitude but opposite directions, either of them is said to be the negative of the other. Thus, in Figure 2,

$$\overrightarrow{AB} = -\overrightarrow{BA}, \text{ and } \overrightarrow{BA} = -\overrightarrow{AB}. \tag{1}$$

The sum of two vectors $\overrightarrow{\mathrm{AB}}$ *and* $\overrightarrow{\mathrm{CD}}$ *is, by definition, the vector* $\overrightarrow{\mathrm{PR}}$

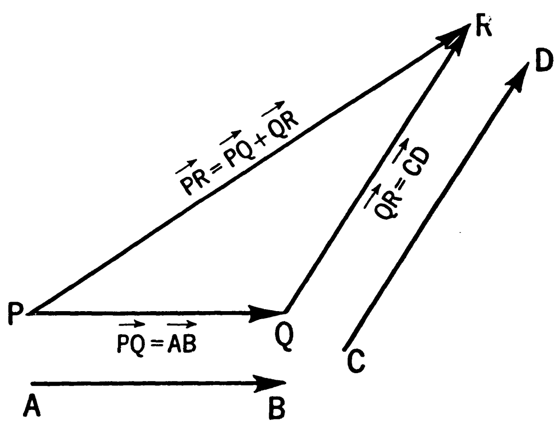


FIG. 3.

obtained as follows: draw from any point P *a vector* $\overrightarrow{\mathrm{PQ}}$ *equal to* $\overrightarrow{\mathrm{AB}}$, *then draw from* Q *a vector* $\overrightarrow{\mathrm{QR}}$ *equal to* $\overrightarrow{\mathrm{CD}}$, *and finally draw*

the sum $\overrightarrow{PR}$ (see Figure 3). If $\overrightarrow{PQ}$ is thought of as representing a first trip and $\overrightarrow{QR}$ a second one, then the sum $\overrightarrow{PQ} + \overrightarrow{QR}$ represents a trip that would bring a person from P to the same point R as the combined trips $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ made continuously and in succession.

Figure 4 shows a vector $\overrightarrow{PS}$ equal to the sum of three vectors $\overrightarrow{AB}$, $\overrightarrow{CD}$, and $\overrightarrow{EF}$; first $\overrightarrow{AB}$ and $\overrightarrow{CD}$ were added to obtain $\overrightarrow{PR}$,

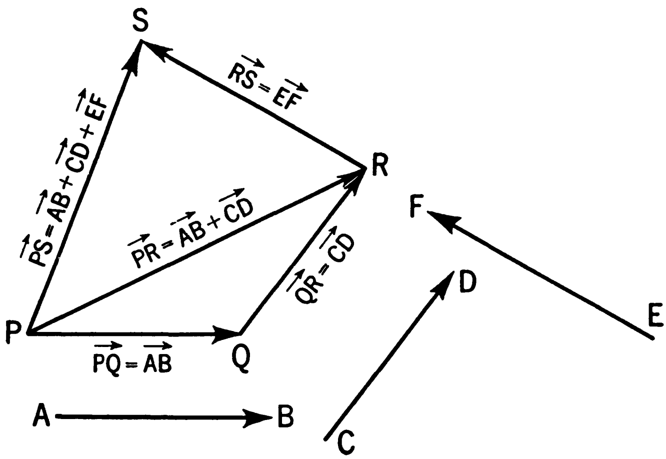


FIG. 4.

and then $\overrightarrow{PR}$ and $\overrightarrow{RS}$ were added to obtain $\overrightarrow{PS}$. Observe, however, that there was no necessity for drawing $\overrightarrow{PR}$.

To subtract a vector $\overrightarrow{CD}$ from a vector $\overrightarrow{AB}$, add the vector $-\overrightarrow{CD} = \overrightarrow{DC}$ to $\overrightarrow{AB}$, as indicated in Figure 5.

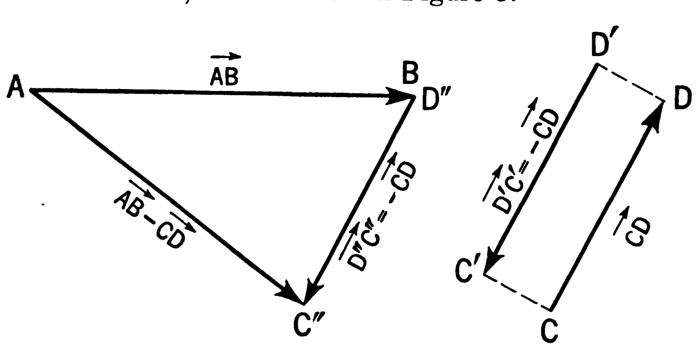


FIG. 5.

The vector equation,

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}, \tag{2}$$

which follows directly from the definition of the sum of two vectors, is extremely important. It applies for the special case in which the three points A, B, and C are in the same straight line as well as for the more general case.

Exercises

1. A man travels east 6 miles and then north 6 miles. Draw two vectors to represent the two trips and add them to obtain a vector representing a single equivalent trip. Give the magnitude and direction of the vector found.

2. Add a vector of magnitude 3 in a horizontal direction and a vector of magnitude 4 in a vertical direction.

3. Draw three vectors to represent three trips, the first 6 miles east, the second 6 miles northeast, and the third 6 miles north, using a ruler and protractor. Add these vectors graphically to obtain a vector representing a resultant trip; measure its magnitude and the angle that its direction makes with east.

4. Draw three vectors to represent three trips made in succession along the same straight line, the first north 10 miles, the second south 6 miles, and the third north 2 miles. Add these vectors and give magnitude and direction of a single equivalent trip.

5. Give the magnitude and direction of the vector obtained by adding a vector $\overrightarrow{AB}$ of 10 units north to a vector $\overrightarrow{CD}$ of 16 units south.

6. Find the sum of the vectors: $\overrightarrow{AB}$ 10 units north, $\overrightarrow{CD}$ 16 units south, $\overrightarrow{EF}$ 25 units north.

7. If "+" means east and "−" west, find the sum of the vectors: $\overrightarrow{AB} = +10$, $\overrightarrow{BC} = +12$, $\overrightarrow{CD} = -42$, $\overrightarrow{DE} = -25$, $\overrightarrow{EF} = +45$.

8. If "+" means northeast and "−" southwest: (a) Find the sum of the vectors: $\overrightarrow{AB} = +16$, $\overrightarrow{BC} = +32$, $\overrightarrow{CD} = -75$, $\overrightarrow{DE} = +x$. Find x when the sum is: (b) Zero. (c) 25 northeast. (d) 60 southwest.

9. Add the three vectors: $\underrightarrow{\ 5\ }$, $\underrightarrow{\ 5\ }$, $\underleftarrow{\ 8\ }$.

10. Add the three vectors: $\overrightarrow{AB} = 10$ northeast, $\overrightarrow{BC} = 10$ southeast, $\overrightarrow{CD} = 10$ east.

11. Add the vectors: $\overrightarrow{AB} = 10$ east, $\overrightarrow{BC} = 20$ north, $\overrightarrow{CD} = 30$ west, $\overrightarrow{DE} = 40$ south.

3. Associating the points on a line with the real numbers

Let the line $X'X$ in Figure 6 extend indefinitely in both directions. Take any two distinct points O and A on the line and write the number zero under O and 1 under A. Take B so that $\overrightarrow{AB} = \overrightarrow{OA}$ and write 2 under B; take C so that $\overrightarrow{BC} = \overrightarrow{OA}$ and write 3 under C; and so on indefinitely. Next take $\overrightarrow{OP} = -\overrightarrow{OA}$ and write -1 under P; take $\overrightarrow{PQ} = -\overrightarrow{OA}$ and write -2 under Q; take $\overrightarrow{QR} = -\overrightarrow{OA}$ and write -3 under R; and so on. In general,

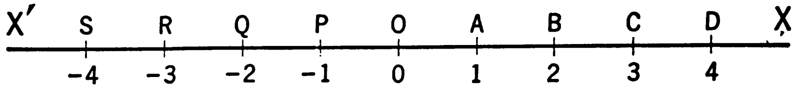


FIG. 6.

if n is any real non-negative number and N a point on $X'X$ such that $\overrightarrow{ON}$ has a magnitude n times that of $\overrightarrow{OA}$, then we assign to N the number n or $-n$ according as $\overrightarrow{ON}$ has the same direction as $\overrightarrow{OA}$ or the opposite direction; or, less accurately stated, to a point N on $X'X$ distant n from O we assign the number $+n$ or $-n$ according as N is on the positive side or the negative side of O.

The point O is called the **origin.** The direction of $\overrightarrow{OA}$ will be called the positive direction of $X'X$ and the direction opposite to that of $\overrightarrow{OA}$ the negative direction of $X'X$. Thus, the positive direction of $X'X$ may be called the direction of increasing associated numbers and the negative direction the direction of decreasing numbers. When we speak of the *direction* of $X'X$, we mean the *positive direction.*

It should be emphasized that these numbers associated with the points on $X'X$ tell not only the distance from O but also the direction from O to the points associated with the numbers. Hence, if x_1 is associated with A_1 and x_2 with A_2, x_1 and x_2 obey the same laws of addition and subtraction as vectors $\overrightarrow{OA_1}$ and $\overrightarrow{OA_2}$. Therefore, from the vector equation

$$\overrightarrow{OA_1} + \overrightarrow{A_1A_2} = \overrightarrow{OA_2}, \tag{3}$$

based on equation (2), we obtain $x_1 + \overrightarrow{A_1A_2} = x_2$, or

$$\overrightarrow{A_1A_2} = x_2 - x_1; \tag{4}$$

this equation holds in magnitude and direction, the direction of the right member being the positive or negative direction of line $X'X$ according as its sign is positive or negative.

The notation $|a|$ will be used to denote the absolute value of a number a. Thus, $|-5| = |5| = 5$, $|-a^2| = a^2$, and, if a is negative, $a = -|a|$. The equation $|1 - x| = 2$ is satisfied if $1 - x = 2$ or -2, that is, if $x = -1$ or 3.

Exercises

1. Draw a straight line, mark on it a point O, and, using O as origin and $\frac{1}{2}$ inch as the unit of measure, plot points on the line corresponding to $1, 3, 2.5, -2, -1.5, \sqrt{2}, -\sqrt{3}$.

2. In Figure 6 how many units apart are the points corresponding to the numbers: (a) 2 and 5? (b) -2 and 3? (c) -4 and 6? (d) m and n? (e) m and $-n$?

3. A vector $\overrightarrow{AK}$ directed along the line $X'X$ of Figure 6 is 2.6 units long. Find the number associated with K if $\overrightarrow{AK}$ has: (a) the same direction as $\overrightarrow{OA}$; (b) the direction opposite to that of $\overrightarrow{OA}$.

4. Replace A by Q in Exercise 3 and solve the resulting problem.

5. If vectors $\overrightarrow{AK}$ and $\overrightarrow{KC}$ lie along the line of Figure 6 and $\overrightarrow{AK}$ is directed positively and has a magnitude of 10 units, find a vector equal to $\overrightarrow{AK} + \overrightarrow{KC}$. Also, if $\overrightarrow{AK}$ has magnitude m, find a vector equal to $\overrightarrow{AK} + \overrightarrow{KC}$.

6. If vectors $\overrightarrow{PK}$, $\overrightarrow{KG}$, and $\overrightarrow{GR}$ lie along line $X'X$ of Figure 6, and if the magnitudes of $\overrightarrow{PK}$ and $\overrightarrow{KG}$ are 6 units and 5 units, respectively, and directed positively, find a vector equal to $\overrightarrow{PK} + \overrightarrow{KG} + \overrightarrow{GR}$. Would the answer be different if different magnitudes were assigned to $\overrightarrow{PK}$ and $\overrightarrow{KG}$?

7. Solve for x: (a) $|2 - x| = 1$. (b) $|3 + x| = 4$. (c) $|2x - 5| = 3$. (d) $|2x + 7| = 0$. (e) $|2x| - 7 = 0$.

4. Plotting points in a plane

Figure 7 shows a line $X'X$ with points associated with numbers by the method used for $X'X$ in Figure 6. Line $Y'Y$ is drawn perpendicular to $X'X$ through O and has its points associated

with numbers in the same manner. Here and elsewhere assume that the basic unit $\overrightarrow{OA'}$ of $Y'Y$ has the same magnitude as $\overrightarrow{OA}$ on $X'X$ unless the contrary is indicated. Thus, any point Q on $Y'Y$ distant q from O is associated with q or $-q$ according as $\overrightarrow{OQ}$ has the same direction as $\overrightarrow{OA'}$ or the opposite direction. Line $X'X$ is called the **x-axis,** $Y'Y$ the **y-axis,** and point O the **origin** of coordinates or simply the **origin.** The four parts into which the plane is divided by the axes, that is, the parts numbered I, II, III, and IV in Figure 7, are called the **first, second, third,** and **fourth quadrants,** respectively.

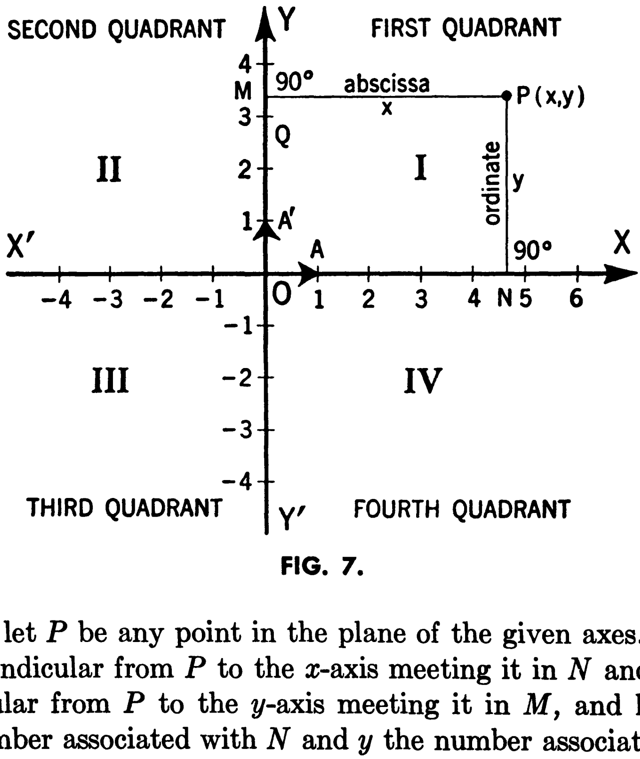


FIG. 7.

Now let P be any point in the plane of the given axes. Draw a perpendicular from P to the x-axis meeting it in N and a perpendicular from P to the y-axis meeting it in M, and let x be the number associated with N and y the number associated with M. Then we call x and y the **coordinates** of point P and designate P by the symbol (x,y). The number x is called the **abscissa** of P and the number y its **ordinate.** Thus, to every point P in the plane there is associated an ordered number pair (x,y), and every ordered pair (x,y) of real numbers determines a unique point, which can be plotted as the intersection of two lines, one drawn perpendicular to the x-axis through the point on it associated

with x and the other perpendicular to the y-axis through the point on it associated with y.

In Figure 8 five points are illustrated. P_1 is represented by (3,2); 3 is the abscissa of P_1 and 2 is its ordinate. P_2 is represented by $(-1,3)$; its abscissa is -1 and its ordinate is 3. Similar statements apply to the points P_3, P_4, and P_5. Throughout the text we shall use the notation $P(x,y)$ to mean **P is identical with (x,y)**. Thus, in Figure 8, point $P_1(3,2)$ may be referred to as P_1 or as (3,2).

Observe in Figure 9 that every point on line GG' has 1 as ordinate. This is expressed otherwise by saying that the equa-

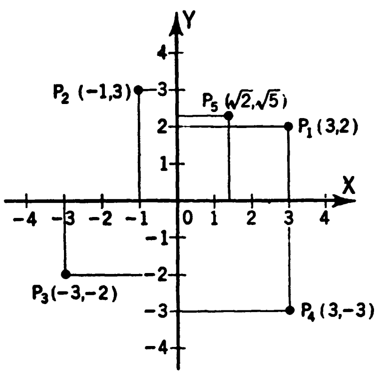


FIG. 8.

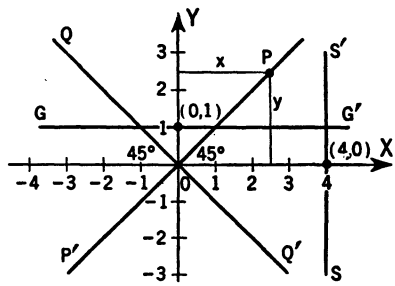


FIG. 9.

tion of GG' is $y = 1$. Similarly, every point on SS' has 4 as abscissa, and we say that the equation of SS' is $x = 4$. Also observe that the abscissa x of any point on $P'P$ is equal to its ordinate y; the equation of line $P'P$ is $x = y$. For every point on line QQ', the abscissa is the negative of the ordinate and the equation of $Q'Q$ is $x = -y$.

Figure 10 represents a piece of coordinate paper. Such paper is generally ruled into squares as indicated. When two perpendicular lines, conveniently placed, are chosen as axes on the coordinate paper, and numbers are placed along them to indicate the association of points to numbers, a point can be readily plotted, approximately, and approximate coordinates of any point are easily read. The student should provide himself with coordinate paper to insure neatness, accuracy, and speed in plotting.

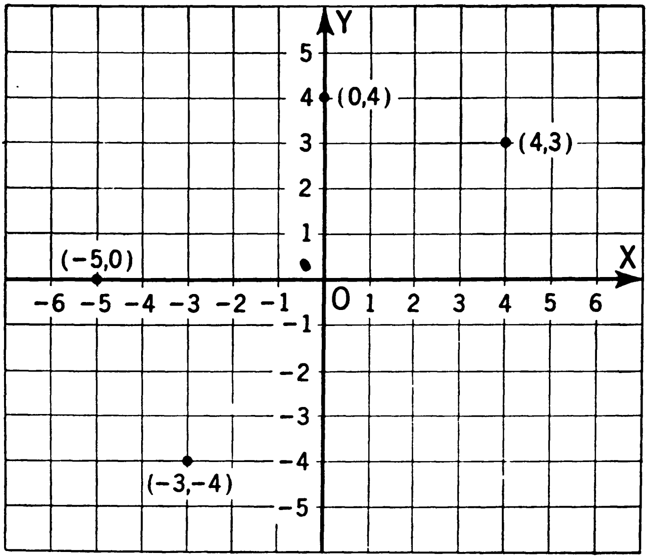


FIG. 10.

Exercises

1. On a piece of coordinate paper mark an x-axis and a y-axis, and write numbers along these axes using a convenient unit. On the paper plot the points (5,2), $(-3,4)$, $(-2,-3)$, $(2,-4)$, (3,0), (0,3), $(0,-2)$, $(-3,0)$, and (0,0).

2. Plot the points (3,0), (3,2), (3,3), $(3,-1)$, $(3,-2)$, and $(3,-3)$. Where do all points lie having as abscissa: (a) 3? (b) 2? (c) -3? (d) 0?

3. Plot the points $(0,-2)$, $(1,-2)$, $(2,-2)$, $(5,-2)$, $(-3,-2)$, and $(-5,-2)$. Where do all points lie having as ordinate: (a) -2? (b) 3? (c) 0?

4. Plot the points (1,1), (2,2), (3,3), $(-1,-1)$, and $(-2,-2)$. Where do all points lie having: (a) abscissa equal to ordinate? (b) abscissa the negative of the ordinate?

5. A vector $\overrightarrow{AB}$ is 6 units long and A is the point (2,3). Find the coordinates of B and of the midpoint on line segment AB if $\overrightarrow{AB}$ has as direction: (a) the positive direction of the x-axis; (b) the negative direction of the x-axis; (c) the positive direction of the y-axis; (d) the negative direction of the y-axis.

6. Replace point (2,3) in Exercise 5 by $(-3,-1)$ and solve the resulting problem.

7. Find the coordinates of the two trisecting points of the line segment connecting: (a) $(1,-3)$ to $(1,9)$; (b) $(3,-4)$ to $(3,6)$; (c) $(-3,-4)$ to $(3,-4)$.

8. Find the coordinates of the two trisecting points of the line segment connecting: (a) $(1,1)$ to $(4,4)$; (b) $(-5,2)$ to $(1,8)$; (c) $(3,-4)$ to $(9,-10)$.

9. Given that the midpoint of vector $\overrightarrow{AB}$ is $(5,6)$, find the coordinates of B if A is: (a) $(-5,3)$; (b) $(-2,-4)$; (c) $(9,7)$.

10. The points $A(0,0)$, $B(4,-2)$, $C(6,1)$ and $D(x,y)$ are the vertices of a parallelogram. Find x and y if a diagonal is: (a) AB; (b) AC; (c) BC.

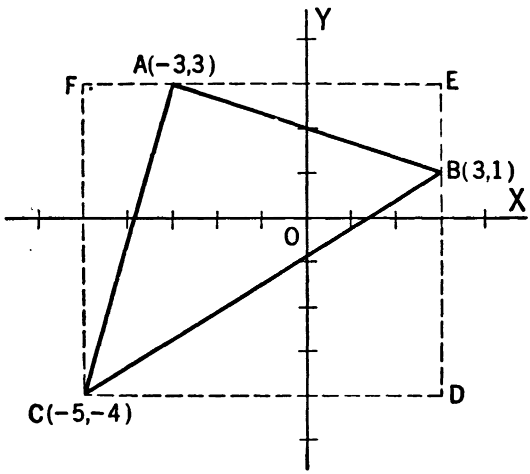


FIG. 11.

11. Find the area of the triangle ABC in Figure 11. To do this find the area of rectangle $CDEF$ and subtract from it the areas of the right triangles CDB, BEA, and AFC.

12. Read Exercise 11 and then find the areas of the triangles having as vertices: (a) $(1,5)$, $(4,4)$, $(0,1)$; (b) $(-4,-2)$, $(0,3)$, $(2,-3)$; (c) $(2,4)$, $(0,-4)$, $(-5,0)$.

CHAPTER II

Graphs of Functions

5. Functions

Many familiar values depend upon other values. The price of a pair of shoes depends, to a certain extent, upon the price of leather. Often the relation is more definite. Thus, there is a relation between the distance fallen by a stone and the time it falls, of pressure in a garden hose and the velocity of the water coming from it, and of the area of an expanding circle and its radius. The word **function** is used in referring to relations of the type mentioned, and the changing quantities are called **variables.** The distance s feet fallen in t seconds by a stone initially at rest is given approximately by

$$s = 16.1t^2. \tag{1}$$

In general, if during a discussion a symbol represents an indefinite number of values, it is called a **variable**; if it represents only one value, it is called a **constant.** The following statement defines a function.

DEFINITION. *If with every value of a first variable* x *in a set of values there is associated one or more values of a second variable* y, *then* y *is a function of* x *for the set.*

For example, when the time is given, the distance fallen by a bomb can be computed by (1). When $t = 1$ second, $s = 16.1 \times 1^2 = 16.1$ feet; when $t = 2$, $s = 16.1 \times 2^2 = 64.4$ feet; and so on. Similarly, from the formula $A = \pi r^2$ for the area A of a circle of radius r, obtain $A = \pi 2^2 = 4\pi$ when $r = 2$; $A = \pi 3^2 = 9\pi$ when $r = 3$; and so on.

6. Functional notation

In the study of analytic geometry, it is convenient to have a simple method of referring to a function under consideration.

Accordingly, such symbols as $f(x)$ and $F(x)$ are used for the purpose when only one variable is involved, and $f(x,y)$ and $F(x,y)$ when two variables are involved. The following examples will give an idea of the implication and use of these symbols.

Example 1. If $f(x) = x^2 + x - 2$, find $f(2)$, $f(3)$, $f(0)$, and $f(y - 2)$.

Solution. $f(2)$ indicates the value found by replacing x by 2 in $f(x)$. Hence, when $f(x) = x^2 + x - 2$,

$$f(2) = 2^2 + 2 - 2 = \mathbf{4}.$$

Similarly,

$$\begin{aligned} f(3) &= 3^2 + 3 - 2 = \mathbf{10}, \\ f(0) &= 0^2 + 0 - 2 = \mathbf{-2}, \\ f(y - 2) &= (y - 2)^2 + y - 2 - 2 \\ &= y^2 - 4y + 4 + y - 4 \\ &= \boldsymbol{y^2 - 3y}. \end{aligned}$$

Example 2. If $f(x,y) = x^2 + y^2 - xy$, find $f(2,3)$, $f(0,-1)$, $f(z - 2,w - 3)$.

Solution. Since $f(2,3)$ indicates the number found by replacing x by 2 and y by 3 in $f(x,y)$, we have, when $f(x,y) = x^2 + y^2 - xy$:

$$\begin{aligned} f(2,3) &= 2^2 + 3^2 - 2 \times 3 = \mathbf{7}, \\ f(0,-1) &= 0^2 + (-1)^2 - 0(-1) = \mathbf{1}, \\ f(z - 2,w - 3) &= (z - 2)^2 + (w - 3)^2 - (z - 2)(w - 3) \\ &= \boldsymbol{z^2 + w^2 - wz - z - 4w + 7}.^* \end{aligned}$$

Exercises

1. If $f(x) = x^2 + 5$, find $f(2)$, $f(3)$, $f(a)$, and $f(-1)$.
2. If $F(y) = y^2 + y - 6$, find $F(2)$, $F(0)$, $F(-2)$, $F(h^2) - F(0)$.
3. If $F(x,y) = x^2 + y^2$, find $F(2,3)$, $F(-1,5)$, $F(0,0)$.
4. If $f(x) = \sin x$, find $f(30°)$, $f(90°)$, $f(0)$, $f(180°)$.
5. If $f(x) = x^2$, find $[f(3x) - f(2x)]/f(x)$.
6. If $f(x) = x^3 - 6x^2 + 12x - 8$, find $f(y + 2)$.
7. If $f(x) = 2x^3 + 6x^2 + 3x - 1$, find $f(y - 1)$.
8. If $f(x, y) = x^2 + y^2 - 2x + 4y + 7$, find $f(x + 1, y - 2)$.
9. If $f(x) = x^2 + x$ and $F(y, z) = y^3 - z^3$, find $f(3) \times F(1,-2)$.

* Answers to illustrative problems will appear in bold-face type throughout the book.

7. Importance of graphs

The student is familiar with various types of graphs exhibiting such things as sizes of countries, exports of various countries, changes in the cost of living, and the rate of growth of population. Observe from the graph of Figure 1, showing the population of the United States by decades, that the population always increased from 1800 to 1930, that the average increase per decade was about 116 million divided by 13 or approximately 9 million per decade, that the increase from 1920 to 1930 was far above average, and that the population has more than doubled since 1890. It is at once apparent that much can be seen from a graph

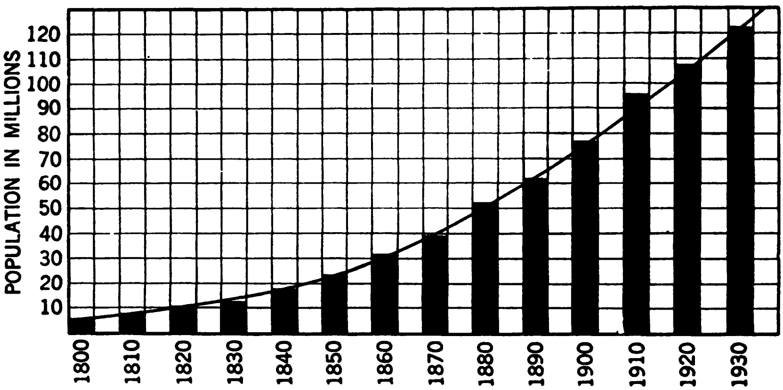


FIG. 1. Graphical Representation of the Growth of Population in the United States.

almost at a glance that would be hidden in a table of statistics. Basically this is the reason why the graphs of analytic geometry are so important. They lend themselves to interpretation; they are highly suggestive of variations and properties deeply hidden in the analytical expressions of functions. Much of this will be apparent as we proceed and the full force of it will appear in the study of calculus, for which analytic geometry prepares the way.

8. Graphs of functions

Consider the function $y(x)$ defined by

$$y = \tfrac{1}{2}x^2. \tag{2}$$

To each value of x it associates the value of y obtained by taking one-half the square of x. Thus, when $x = 2$, $y = \frac{1}{2}\,2^2 = 2$; when $x = 3$, $y = \frac{1}{2}\,3^2 = 4.5$; and so on. The following table shows some corresponding values of x and y obtained from (2):

x	-3	-2	-1	0	1	2	3
y	4.5	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	4.5

(3)

Plotting the points having corresponding pairs of numbers as coordinates, we obtain the points enclosed in small circles in Figure 2. The smooth curve drawn through these points represents the graph of $y = \frac{1}{2}x^2$. The graph thus obtained is an imperfect representation of the true graph defined below. Nevertheless, we see from it that the value of the function is least

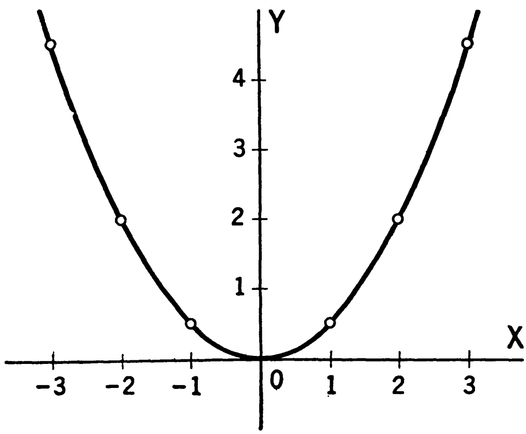


FIG. 2. Graph of $y = \frac{1}{2}x^2$.

when $x = 0$; that as x increases numerically, so does y and at an ever-increasing rate; and that the symmetry of the function is suggested by the symmetry of the graph with respect to the y-axis.

Frequently a point (x,y) is spoken of as *satisfying an equation.* A point satisfies an equation in x and y if the numbers obtained from its members by replacing x by the abscissa of the point and y by its ordinate are equal. Thus, $(-2,2)$ satisfies equation (2), since $2 = \frac{1}{2}(-2)^2$.

DEFINITION. *A curve is the graph of an equation in* x *and* y *if: (a) every point on the curve satisfies the equation; (b) every point which satisfies the equation lies on the curve.*

For example, if the graph of Figure 2 were perfect, every point on it would satisfy equation (2) and every point represented by value pairs in table (3) would lie on the graph.

The graph of Figure 2 is imperfect for three reasons. First, it is a narrow band instead of a curve having no width. Second, except for the actual points plotted, it is only approximate; if we should measure carefully the coordinates of a point on it, they would satisfy (2) only approximately; and, finally, it does not satisfy part (b) of the definition. For example, the point (50,1250) satisfies (2) but this point does not lie on the graph shown in Figure 2. The true graph of (2) has two branches extending farther and farther without end. Although every curve we draw will have limitations of the kind mentioned, we shall speak of our imperfectly drawn curves as *graphs*, and shall speak of the process of constructing them as *graphing;* but we shall realize that the true graph fulfills the conditions of the definition exactly. Also, for convenience, we will often speak of an *equation* and its *graph* interchangeably. For example, we shall speak of "the curve $x^2 + y^2 = 25$" instead of "the curve represented by $x^2 + y^2 = 25$." The following rule may be used in drawing the graph of a curve.

RULE: *To plot or graph an equation in the variables* x *and* y:

(a) *Solve the given equation for one variable in terms of the other, say,* y *in terms of* x.

(b) *Substitute values for* x, *find the corresponding values of* y, *and tabulate the results. Use enough points to bring out essential features.*

(c) *Plot the points represented by the number pairs in the table and connect them by a smooth curve.*

Example 1. Plot $x^3 - y = 3x$.

Solution. The solution of the given equation for y is

$$y = x^3 - 3x.$$

Substituting values of x from -3 to 3 in the given equation and computing the corresponding values of y, we obtain the following table:

x	-3	-2	-1	0	1	2	3
y	-18	-2	2	0	-2	2	18

Plotting the points represented by the number pairs of the table and connecting them with a smooth curve, we obtain the graph shown in Figure 3. The true graph extends indefinitely up and

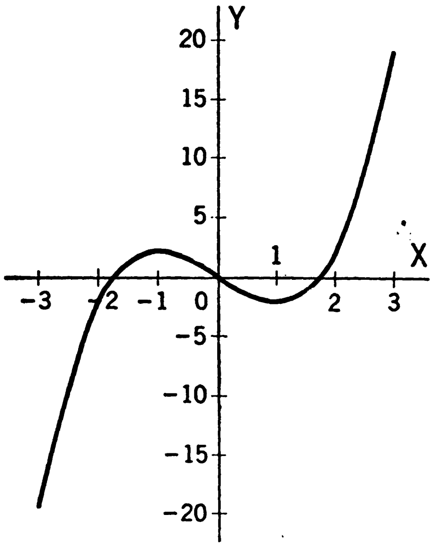


FIG. 3. Graph of $x^3 - y = 3x$.

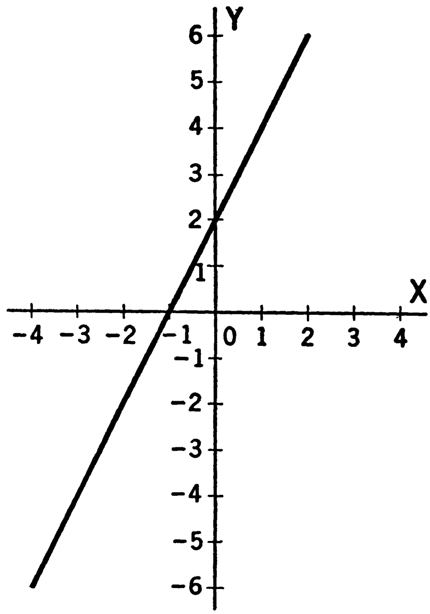


FIG. 4.

down. Observe that a different unit of measure was used on the y-axis from that used on the x-axis. This was done to obtain compactness and good appearance.

Example 2. Graph the equation $y - 2x = 2$.

Solution. Here $y = 2x + 2$. For values of x from -4 to 2 we obtain the following table:

x	-4	-3	-2	-1	0	1	2
y	-6	-4	-2	0	2	4	6

Plotting the points from the number pairs in the table and connecting them smoothly, we obtain the graph shown in Figure 4. Observe that it is a straight line. It will be shown in §23 that *the graph of an equation of the first degree in* x *and* y *is a straight line.*

Exercises

1. Sketch the graph of $y = \frac{1}{4}x^2$ for x ranging from -4 to $+4$.

2. Taking account of the italicized definition, determine whether each point lies on the graph of the associated equation:

(a) $y = 3x - 7$; $(2,3)$.
(b) $y = 4x + 10$; $(2,-3)$.
(c) $x^2 + y^2 = -3$; $(-2,1)$.
(d) $3x^2 + 4y^2 = 4$; $(1,-\frac{1}{2})$.
(e) $y = \sin x$; $(\pi,0)$.
(f) $y = 3^x$; $(2,9)$.
(g) $y = \sqrt{x}$*; $(4,-2)$.
(h) $y = \frac{1}{4}x^2$; $(21,110)$.

3. Plot the points $(5,1)$, $(5,2)$, $(5,3)$, $(5,-1)$, and $(5,-2)$, and then graph $x = 5$. Also graph $y = 3$.

Plot the graphs of the equations numbered 4 to 9. The tables of corresponding values should cover at least the indicated ranges for x.

4. $y = 2x - 3$; -2 to 4.

5. $y = x^2 - 3x$; -2 to 5.

6. $y = x^3 - 9x$; -4 to 4. Let each space along the x-axis represent 1 unit and each space along the y-axis, 5 units.

7. $y = 4x/(x^2 + 1)$; -4 to 4.

8. $y = x^2/(x^2 + 1)$; -5 to 5.

9. $x + 2y = 6$; -2 to 8.

10. For each equation determine k so that the graph of the equation will contain the indicated point:

(a) $x^2 + y^2 = k$, $(2,4)$.
(b) $x^2 + kx = y^2$, $(-2,-4)$.
(c) $3xy + ky^2 = kx + 6$, $(2,-3)$.
(d) $x + y - 3 + k(2x - y + 4) = 0$, $(0,-2)$.

11. Show that the graph of $x^2 + y^2 = 100$ meets the graph of $y = x + 14$ in $(-8,6)$ and in $(-6,8)$.

Hint. Show that each point lies on both graphs.

12. Does the graph of $(x - y)(x^2 + y^2 - 25) = 0$ contain each of the following points: $(1,1)$, $(2,2)$, (m,m), $(3,4)$, $(3,-4)$, $(\sqrt{21},2)$, $(5,0)$, $(m,\sqrt{25 - m^2})$ where $m < 5$? If the coordinates of a point satisfy either one of the equations $x - y = 0$ or $x^2 + y^2 - 25 = 0$, will the point lie on the graph of $(x - y)(x^2 + y^2 - 25) = 0$?

13. Show that $(5,-6)$ lies on the graph of $x^2 + y^2 = 61$ but not on the graph of $y = \sqrt{61 - x^2}$.† Draw the graph of $x^2 + y^2 = 61$ and of $y = \sqrt{61 - x^2}$.

---

* See the footnote to Exercise 13 below.

† By convention, $\sqrt{4} = 2$, not -2. To indicate that both signs are to be considered, we must write $\pm\sqrt{4}$. Therefore, $(6,5)$ satisfies $x^2 + y^2 = 61$ but not $y = -\sqrt{61 - x^2}$.

14. Does the graph of $x - 2\sqrt{x} = 3y$ contain the point: (a) (9,1)? (b) (9,5)? Does the graph of $(2\sqrt{x})^2 = 4x = (x - 3y)^2$ contain the point: (c) (9,1)? (d) (9,5)? (e) When a radical is removed from a first equation in x and y by squaring both members to obtain a second equation, does the graph of the first equation necessarily contain the graph of the second? Does the graph of the second equation necessarily contain that of the first?

15. Show that the point $(2,-3)$ satisfies $(x + y + 1) + k(2x + y - 1) = 0$ for all values of k.

16. Show that points (3,4) and (5,0) satisfy $(x^2 + y^2 - 25) + k(2x + y - 10) = 0$ for all values of k.

17. Graph $y^2 = x$ and $y = \sqrt{x}$.

18. Graph $y = 2 + \sqrt{x}$ and $(y - 2)^2 = x$.

9. *Intercepts, symmetry, and extent*

The process of graphing an equation is often simplified by a preliminary discussion relating to the concepts of intercepts, symmetry, and extent.

Intercepts. The x-*intercepts* of a curve are the abscissas of the points where the curve crosses or touches the x-axis, and the y-*intercepts* are values of y at points where the curve crosses or touches the y-axis. The x-intercepts are found by setting $y = 0$ *in the given equation and solving* the result for x, and the y-intercepts are found by setting $x=0$ in the equation and solving the result for y. For example, to find the intercepts of the curve having $4x^2 + y^2 = 36$ as equation, set $y = 0$ in it to obtain $4x^2 = 36$, or $x = \pm 3$ as the x-intercepts; and set $x = 0$ in $4x^2 + y^2 = 36$ to obtain $y^2 = 36$, or $y = \pm 6$ as the y-intercepts.

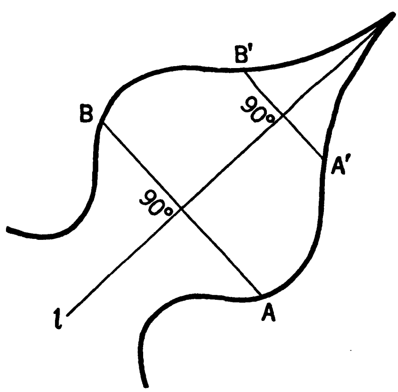


FIG. 5.

Symmetry. *Two points* A *and* B *are symmetric to a line* l *if* l *is the perpendicular bisector of segment* AB. Thus, A and B in Figure 5 are symmetric to l; and in Figure 6, P and Q are symmetric to the x-axis and P and M are symmetric to the y-axis.

A curve C *is symmetric to a line* l *if every point on* C *but not on* l *is one of a pair of points on* C *symmetric to* l. Thus, in Figure 5 curve $AA'BB'$ is symmetric to l, and the circle in Figure 7 is symmetric to the x-axis and to the y-axis.* *Two points* A *and* B *are symmetric to a point* P *if* P *is the midpoint of segment* AB. Thus points P and N in Figure 6 are symmetric to O. *A curve* C *is symmetric to a point* P *if every point on* C, *except possibly* P, *is one of a pair of points on* C *symmetric to* P. Thus, the circle in Figure 7 is symmetric to O.

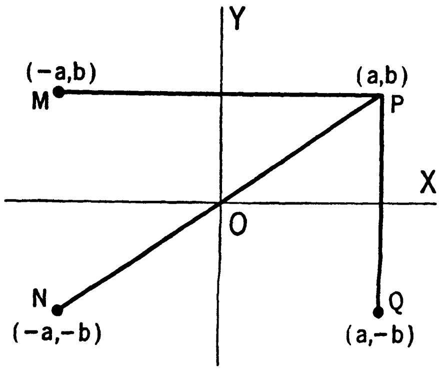


FIG. 6.

If, when x *is replaced by* $-$x, *the same or an equivalent*† *equation results, the graph of the equation is symmetric to the* y-*axis.* For, if either of the pair of points (a,b) and $(-a,b)$, symmetric to the y-axis, satisfies the equation, then the other must satisfy it. Similarly, *if the same or an equivalent*† *equation is obtained when* y *is replaced by* $-$y, *the graph of the equation is symmetric to the* x-*axis.* Also, *if the same or an equivalent*† *equation results when* x *is replaced by* $-$x *and* y *by* $-$y, *the graph of the equation is symmetric to the origin.* For if either of the pair of points (a,b) and $(-a,-b)$ satisfies the equation, then the other must satisfy it also.

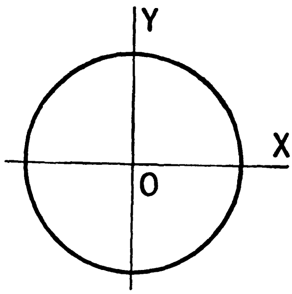


FIG. 7.

From the italicized statements made above, it follows that: *an equation containing only even powers of* x *has a graph symmetric to the* y-*axis; an equation containing only even powers of* y *has a graph symmetric to the* x-*axis; an equation containing only terms of even degree, or only terms of odd degree in* x *and* y, *is symmetric to the origin.*

* Put a drop of ink on a piece of paper, fold the paper, and press it smooth. Then return the paper to its original shape and you will have a figure symmetric to the crease along which the paper was folded.

† Two equations are here called equivalent if either may be obtained from the other by multiplying it through by 1 or -1.

Extent. An even root of a negative number is an imaginary number. In this book, only points having both coordinates real will be considered. Thus, from the equation $x^2 + y^2 = 25$, or $y = \pm\sqrt{25 - x^2}$, it appears that if x_1 is a number numerically greater than 5, then $y = \pm\sqrt{25 - x_1^2}$ is imaginary. If $x_1 = 6$, for example, $y = \pm\sqrt{25 - 36} = \pm\sqrt{-11}$. Here the curve is plotted only for x in the range from -5 to 5 inclusive. Figure 8 shows the graph of $x^2 + y^2 = 25$. *To discuss an equation for extent of graph, solve the equation for* y *in terms of* x *and from the result find the values of* x *that lead to an even root of a negative number.* The graph does not exist for these values. Likewise, *consider the result of solving the equation for* x *in terms of* y *to get the extent as regards values of* y.*

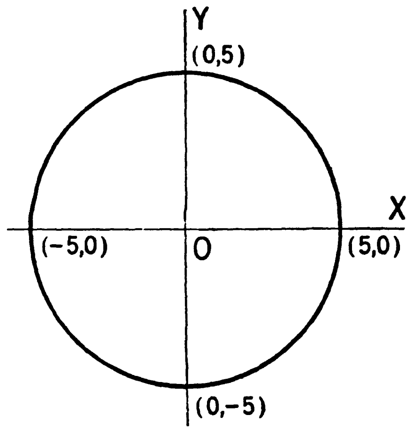


FIG. 8.

Example 1. Graph the ellipse† represented by $x^2 + 4y^2 = 16$.

Solution. Replacing x by zero in $x^2 + 4y^2 = 16$, we obtain as the y-intercepts $y = \pm 2$. Replacing y by zero, we find as x-intercepts $x = \pm 4$.

The curve is symmetric to the x-axis, the y-axis, and the origin. Since, from the given equation,

$$y = \pm\tfrac{1}{2}\sqrt{16 - x^2}, \tag{a}$$

x cannot be greater than 4 numerically. Likewise, from

$$x = \pm 2\sqrt{4 - y^2} \tag{b}$$

we deduce that y must be less than or equal to 2 numerically.

---

* The extent of a graph may be limited by considerations other than those considered here. For example, $y = \sin x$ exists only on or between the lines $y = 1$ and $y = -1$, and $y = 10^x$ is never negative.

† An ellipse is a curve representable by an equation having the form $b^2x^2 + a^2y^2 = a^2b^2$, where $a \neq 0$, $b \neq 0$.

From this information regarding intercepts, symmetry, and extent, and the table of values obtained from (a),

x	0	1	2	3	4
y	2	$\frac{1}{2}\sqrt{15} = 1.94$	$\frac{1}{2}\sqrt{12} = 1.73$	$\frac{1}{2}\sqrt{7} = 1.32$	0

we plot the ellipse shown in Figure 9.

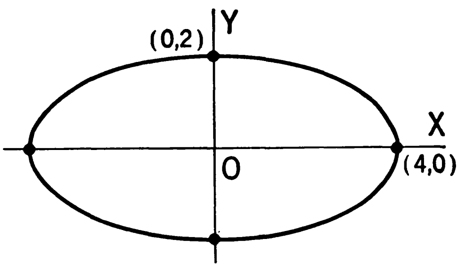


FIG. 9. Graph of $x^2 + 4y^2 = 16$.

Example 2. Graph the parabola* represented by $y^2 = 4 - x$.

Solution. Setting $x = 0$ and $y = 0$ in turn in $y^2 = 4 - x$, we obtain:

$$y\text{-intercepts} = \pm 2,$$
$$x\text{-intercept} = 4.$$

The curve is symmetric to the x-axis.

Since $y = \pm\sqrt{4 - x}$, the graph does not exist if $4 - x$ is negative, that is, if $x > 4$. There is no limit on the values of y.

Using this information and the table of values computed from $y = \sqrt{4 - x}$,

x	4	3	2	1	0	-1	-2	-3
y	0	1	1.41	1.73	2	2.23	2.45	2.64

we obtain the graph shown in Figure 10. The true curve extends leftward without end.

Exercises

1. Find the x-intercepts and the y-intercepts of the curves represented by:

(a) $x^2 + y^2 = 25$.
(b) $x^2 + 4y^2 = 4$.
(c) $x^2 - 4y^2 = 16$.
(d) $x^2 + 2x + y^2 = 35$.
(e) $y^2 = (x + 1)(x - 1)(x - 4)$.
(f) $y^2 + 6x - x^2 = 9$.

* A parabola is a curve representable by an equation having the form $y^2 = ax + b$, where $a \neq 0$.

2. Discuss the graphs represented by the following equations as regards symmetry to the x-axis, the y-axis, and the origin:

(a) $3x^2 + 4y^2 = 25.$
(b) $x^2 - 4y^2 = 16.$
(c) $x^2 = 2y.$
(d) $y = ax^3 + bx.$
(e) $x^3y = x + y^3.$
(f) $(x^2 - y^2)^2 = x^2 + y^2.$
(g) $y = |x|.$
(h) $|y| = |x|.$
★(i) $y^3 = |x|.$*

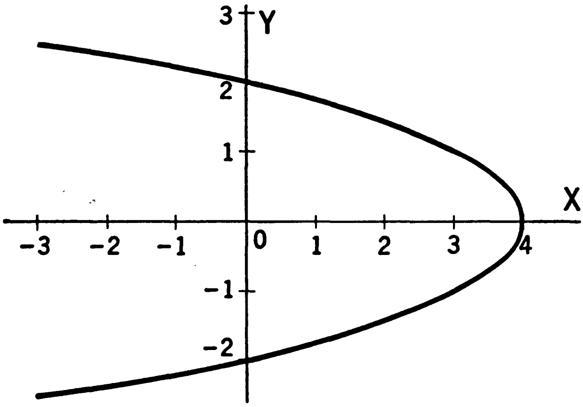


FIG. 10. Graph of $y^2 = 4 - x$.

3. For what values of x and of y are there no points on the graphs of the following equations:

(a) $y^2 = 4 + x.$
(b) $x^2 + 4y^2 = 16.$
(c) $4y^2 - x^2 = 16.$
(d) $x^2 + y^2 + 9 = 0.$
(e) $3x^2 + 4 - 4y^2 = 0.$
(f) $(x - 2)^2 + y^2 = 0.$
(g) $x^3 = y^2 - 8.$
(h) $x^2 + y^2 = 3x.$

4. If A,B, and C mean symmetry of a given curve with respect to the x-axis, the y-axis, and the origin, respectively: (a) Do A and B imply C? (b) Does C imply both A and B? (c) Do A and C imply B?

5. If a curve is represented by a polynomial equated to zero, all of the terms of the polynomial being of odd degree in x and y, show that the curve is symmetric to the origin.

Discuss, as regards symmetry, intercepts, and extent, and sketch the graphs of the equations numbered 6 to 21:

6. $2y = 7x - 14.$
7. $4x^2 + 9y^2 = 36.$
8. $y^2 = 4x.$
9. $4x^2 - y^2 = 4.$†
10. $y^2 = x^3.$
11. $y^2 = x + 4.$
12. $x^2 = 4 - y.$
13. $10y = x^3.$

* A problem marked with a star ★ is either difficult or long.

† A curve representable by an equation having the form $b^2x^2 - a^2y^2 = a^2b^2$, $a \neq 0$, $b \neq 0$, is called a **hyperbola**.

14. $10y = x^3 - 4x$.

15. $10y^2 = x(x - 1)(x - 4)$.

★16. $y = |x|$.

★17. $|y| = x$.

★18. $|y| = |x|$.

★19. $y = x^2|x|$.

★20. $|y| = x^2 - 2x$.

★21. $y = x^2 - 2|x|$.

10. Translation of axes

If a circle is referred to axes through its center, it has the equation $x^2 + y^2 = a^2$; when referred to axes neither of which contains the center, its equation has the form $x^2 + y^2 + Ax + By + C = 0$. The method of this article is used to obtain, from given equations, simpler equations.

Figure 11 shows a point with coordinates (x,y) referred to

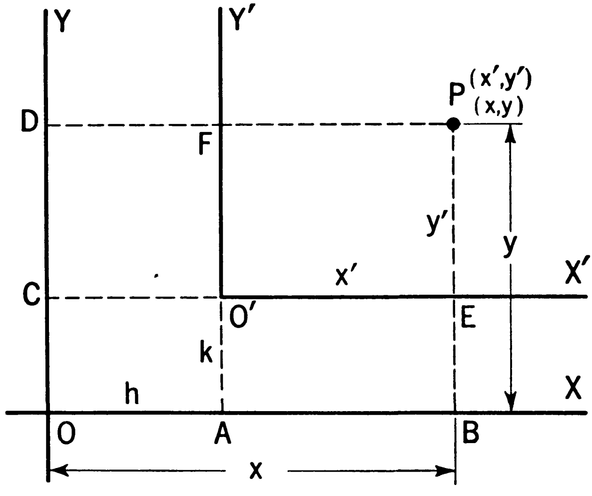


FIG. 11.

axes OX and OY, and with coordinates (x',y') referred to the parallel axes $O'X'$ and $O'Y'$, where O' is the point (h,k). Taking account of equation (4), §3, and the remarks in the paragraph containing this equation, we have

$$x = \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OA} + \overrightarrow{O'E} = h + x' \tag{4}$$

$$y = \overrightarrow{OD} = \overrightarrow{AO'} + \overrightarrow{O'F} = \overrightarrow{AO'} + \overrightarrow{EP} = k + y', \tag{5}$$

or

$$\begin{aligned} \mathbf{x} &= \mathbf{x}' + \mathbf{h} \\ \mathbf{y} &= \mathbf{y}' + \mathbf{k}. \end{aligned} \tag{6}$$

The following examples will show the usefulness of these formulas. The process of using (6) to derive from an equation a simpler equation is called **translation of axes.**

Example 1. Sketch the graph of the equation

$$(x - 2)^2 + (y - 3)^2 = 9. \qquad \text{(a)}$$

Solution. Equations (6) for axes parallel to the original ones and with new origin (2,3) are

$$x = x' + 2,\ y = y' + 3. \qquad \text{(b)}$$

Substituting these values of x and y in (a), obtain

$$(x' + 2 - 2)^2 + (y' + 3 - 3)^2 = 9,$$
$$\text{or } x'^2 + y'^2 = 9. \qquad \text{(c)}$$

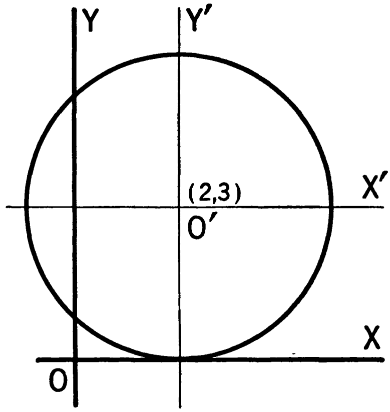


FIG. 12

Drawing both sets of axes and sketching the graph relative to the x', y'-axes, without reference to the old ones, we obtain Figure 12.

Example 2. Sketch the graph of

$$4x^2 + 9y^2 - 16x + 18y = 11. \qquad \text{(a)}$$

Solution. Substituting x and y from (6) in (a), and collecting coefficients of like terms, obtain

$$4x'^2 + 9y'^2 + (8h - 16)x' + (18k + 18)y' + 4h^2$$
$$+ 9k^2 - 16h + 18k = 11. \qquad \text{(b)}$$

Now h and k are at our disposal. Choose h and k so that

$$8h - 16 = 0,\ 18k + 18 = 0, \quad \text{or} \quad h = 2,\ k = -1. \qquad \text{(c)}$$

Substituting these values for h and k in (b), obtain

$$4x'^2 + 9y'^2 = 36. \qquad \text{(d)}$$

Now draw both sets of axes and sketch the graph from (d) on the x', y'-axes to obtain the graph of Figure 13 on the next page.

Exercises

Using formula (6) and the indicated new origin, find simple equations from the following equations numbered 1 to 5:

1. $(x - 3)^2 - (y - 2)^2 = 4$; (3,2).

2. $(y + 2)^2 = 4x - 12$; (3,−2).

3. $(x - 1)^2 + 4(y + 2)^2 = 4$; $(1,-2)$.

4. $x^2 + 2y^2 + 4x = 0$; $(-2,0)$.

5. $y = x^3 + 3x^2 + 4x + 5$; $(-1,3)$.

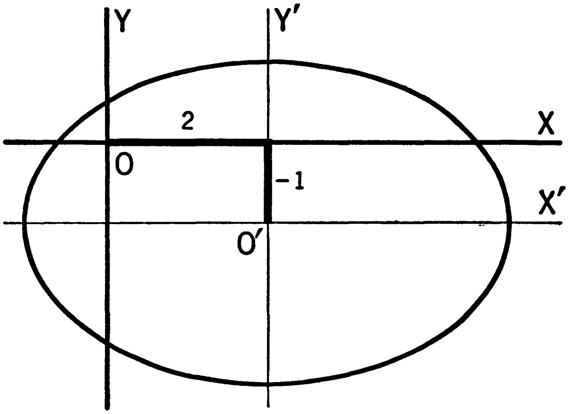


FIG. 13.

Proceed as in Example 2 of this article to derive from each of the equations numbered 6 to 13 another equation having the form written to the right of it. Then draw both sets of axes and sketch the curve from the derived equation:

6. $x^2 + 2x + y^2 - 4y = 4$; $x'^2 + y'^2 = A$.

7. $4x^2 - 8x + y^2 + 4y = 1$; $4x'^2 + y'^2 = A$.

8. $y^2 + 6x - 4y + 10 = 0$; $y'^2 = Ax'$.

9. $4x^2 + 12x - 20y + 49 = 0$; $4x'^2 = Ay'$.

10. $4x^2 - y^2 - 4x + 6y = 12$; $4x'^2 - y'^2 = A$.

11. $y = x^3 - 3x^2 - x + 4$; $y' = x'^3 + Ax'$.

12. $y = 2x^3 - 18x^2 + 46x - 32$; $y' = Ax'^3 + Bx'$.

13. $(y - 2)^2 = (x + 1)^3$; $y'^2 = x'^3$.

Simplify each of the equations numbered 14 to 19 by translation of axes, and find from the result axes or points of symmetry of the corresponding graph:

14. $2(x - 2)^2 + (y + 3)^2 = 2$.

15. $4(x + 3)^2 - y^2 = 4$.

16. $(x + 2)(y + 3) = 10$.

17. $y = (x + 1)^3$.

18. $x^2 + 2x + 4y^2 = 8$.

19. $4x^2 + 8x + y^2 - 6y = 3$.

★**20.** By translation of axes, transform $y = Ax^3 + Bx^2 + Cx + D$, $A \neq 0$, to the form $y' = Ax'^3 + [(3AC - B^2)/(3A)]x'$, and thus show that its graph always has a point of symmetry.

11. Composite graphs

The graph (see Figure 14) of

$$(x - y)(x^2 + y^2 - 25) = 0 \tag{7}$$

consists of the graphs of

$$x - y = 0 \text{ and } x^2 + y^2 - 25 = 0. \tag{8}$$

For, any point (a,b) on the graph of either equation of (8) will satisfy (7), since the product $(a - b)(a^2 + b^2 - 25)$ is zero if either factor is zero. Also, if any point (a,b) satisfies (7), then $(a - b)(a^2 + b^2 - 25) = 0$, and this can be true only if one of its factors is zero; it follows that any point on the graph of (7)

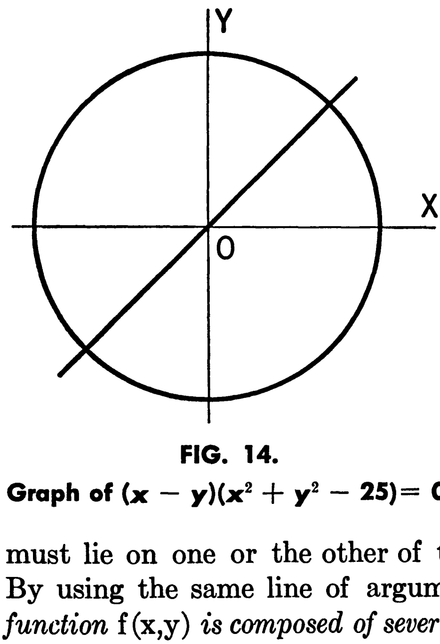


FIG. 14.
Graph of $(x - y)(x^2 + y^2 - 25) = 0$.

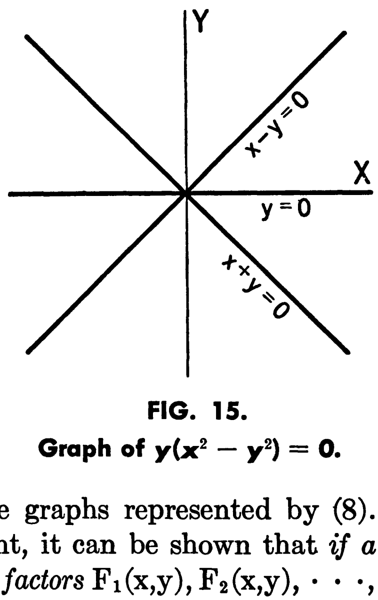


FIG. 15.
Graph of $y(x^2 - y^2) = 0$.

must lie on one or the other of the graphs represented by (8). By using the same line of argument, it can be shown that *if a function* f(x,y) *is composed of several factors* $F_1(x,y), F_2(x,y), \cdots,$ *then the graph of*

$$f(x,y) = F_1(x,y)\, F_2(x,y) \cdots = 0$$

consists of the combined graphs of

$$F_1(x,y) = 0,\ F_2(x,y) = 0,\ \cdots.$$

For example, the graph of $y(x^2 - y^2) = 0$ is the combined graphs of

$$y = 0,\ x + y = 0, \text{ and } x - y = 0$$

as shown in Figure 15.

The principle just explained enables one to see the relation between the graph of a given equation and the graph of an equation obtained from the given one by squaring. Any point satisfying $f^2(x,y) = F^2(x,y)$ satisfies either $f(x,y) = F(x,y)$ or $f(x,y) = -F(x,y)$, and conversely. For example, the graph of $y^2 = 16 - x^2$ contains all points satisfying either $y = \sqrt{16 - x^2}$ or $y = -\sqrt{16 - x^2}$, and conversely; also all points satisfying either $\sqrt{(x-2)^2 + y^2} = y + 2$ or $\sqrt{(x-2)^2 + y^2} = -y - 2$ will lie on the graph of $(x-2)^2 + y^2 = (y+2)^2$, and conversely.

Exercises

Sketch the graphs represented by the equations numbered 1 to 11:

1. $(x + 3y)(x^2 - y^2) = 0.$

2. $xy(x + y - 5) = 0.$

3. $(x + y - 5)(x - y + 5) = 0.$

4. $xy + y^2 - 2y = 0.$

5. $(x - 3)(x^2 + y^2 - 16) = 0.$

6. $x^3 - 4xy^2 = 0.$

7. $(x - y)(x^2 + 2y) = 2(x - y).$

8. $y^3 = y(4 - x).$

9. $x^2 - y^2 + 7(x + y) = 0.$

10. $x^2 - 2xy - 3y^2 = 0.$

11. $x^2 - 2xy + y^2 + 2(x - y) = 0.$

Graph each equation of the pairs of equations numbered 12 to 16 and state the relation between the graphs:

12. $y = \sqrt{a^2 - x^2}$, $y^2 = a^2 - x^2$.

13. $x = \sqrt{4 - y^2}$, $x^2 = 4 - y^2$.

14. $y + \sqrt{x} = 6$, $x = (6 - y)^2$.

15. $x = \sqrt{y^2 - 1}$, $x^2 = y^2 - 1$.

16. $y = 2 + \sqrt{9 - x^2}$, $(y - 2)^2 = 9 - x^2$.

12. Points of intersection of graphs

Since a point of intersection of two graphs lies on both of them, its coordinates, in accordance with the definition of §8, satisfy the equations of both graphs and are therefore a simultaneous

solution of the equations. Conversely, a simultaneous solution of two equations satisfies both of them; therefore, in accordance with the definition of §8, a real simultaneous solution lies on both of their graphs and is a point of intersection of the graphs. The student will get some idea of the wide application of this principle from the following examples and exercises.

Example 1. Find the points of intersection of the graphs of $x^2 + y^2 = 13$ and $x - 2y + 1 = 0$.

Solution. Solve the second given equation for x and substitute in the first to obtain

$$(2y - 1)^2 + y^2 = 13.$$

Proceed as indicated in the following equations:

$5y^2 - 4y - 12 = 0$,

$(5y + 6)(y - 2) = 0, y = 2, y = -\frac{6}{5}$.

Substitute these values in $x - 2y + 1 = 0$ to obtain

$x = 2y - 1 = 4 - 1 = 3$ when $y = 2$,

$x = 2y - 1 = -\frac{12}{5} - 1 = -\frac{17}{5}$ when $y = -\frac{6}{5}$.

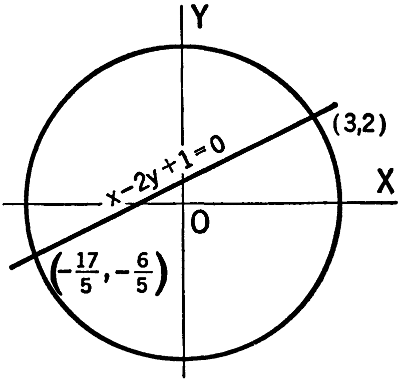


FIG. 16.

Hence, the required points are (3, 2), $(-\frac{17}{5}, -\frac{6}{5})$. Figure 16 represents the situation graphically.

Example 2. Show analytically that the line $x + y = 8$ does not intersect the curve $x^2 + y^2 = 13$.

Solution. Replacing y in $x^2 + y^2 = 13$ by $8 - x$ from $x + y = 8$, and solving the resulting quadratic equation, we get

$$x^2 + (8 - x)^2 = 13,$$

$$2x^2 - 16x + 51 = 0,$$

$$x = \frac{16 \pm \sqrt{256 - 8(51)}}{4}$$

$$= 4 \pm \tfrac{1}{2}\sqrt{-38}.$$

Since the only simultaneous solutions are imaginary, and since imaginary numbers are not represented on our graphs, the graphs have no points of intersection.

Example 3. (a) Show that a curve represented by

$$(x^2 + y^2 - 13) + k(x - 2y + 1) = 0, \tag{I}$$

where k represents any constant, contains the points of intersection of the graphs of

$$x^2 + y^2 - 13 = 0 \text{ and } x - 2y + 1 = 0. \tag{II}$$

(b) Determine k if the graph of equation (I) passes through (0,0).

Solution. (a) Let (m,n) be any point of intersection of the graphs represented by (II). Then,

$$m^2 + n^2 - 13 = 0, m - 2n + 1 = 0. \tag{III}$$

(Why?) Now substitute (m,n) in (I) to obtain

$$(m^2 + n^2 - 13) + k(m - 2n + 1) = (0) + k(0) = 0.$$

Hence, (m,n) lies on the graph of (I). (Why?) Hence, the graph of (I) contains all points of intersection of the graphs represented by (II).

(b) Since (0,0) is to lie on the graph of (I), the coordinates (0,0) must satisfy (I). Hence,

$$(0 + 0 - 13) + k(0 - 0 + 1) = 0, \text{ and } k = \mathbf{13}.$$

Exercises

Find the points of intersection of the graphs of the pairs of equations numbered 1 to 8. In each case, draw a sketch showing graphs and points of intersection.

1. $\begin{cases} x + 2y = 3, \\ x - 2y = 15. \end{cases}$

2. $\begin{cases} 3x + 2y = 7, \\ 2x - 3y = -4. \end{cases}$

3. $\begin{cases} x^2 + 2y^2 = 48, \\ x - y = 0. \end{cases}$

4. $\begin{cases} x^2 - y^2 = 16, \\ y = x - 8. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 25, \\ 4x^2 + y^2 = 52. \end{cases}$

6. $\begin{cases} y^2 = x + 3, \\ x^2 + 4y^2 = 9. \end{cases}$

7. $\begin{cases} y = x^3 - x, \\ y = 3x. \end{cases}$

8. $\begin{cases} y^2 = x + 2, \\ y^2 = 2x. \end{cases}$

9. Show that the graph of $4x^2 + y^2 = 20$ is met by the graph of: (a) $x + y = 0$ in two points; (b) $x + y = 5$ in one point only; (c) $x + y = 6$ in no points. Sketch all graphs involved.

10. The graphs of $mx + y = 1$ and $2x + 3y = n$ meet in $(1,-2)$. Find m and n.

11. Find k if the graph of $y = x^2$ meets the graph of $y + 4x = k$ in $(-5,25)$ and then find another point in which the graphs intersect.

12. Find the point of intersection of the graphs of $x + y = 8$ and $2x - y = 10$; show that this point lies on $(x + y - 8) + k(2x - y - 10) = 0$ no matter what value k represents, and determine k if this last equation is satisfied by $(1,1)$.

13. To find the points of intersection of the graphs of

$$(x + y)(x^2 + y^2 - 25) = 0 \text{ and } 2x - y = 2,$$

set each factor of the first equation equal to zero, and solve each of the resulting equations simultaneously with the second. Check your points by comparing with roughly estimated coordinates of points A,B, and C in Figure 17.

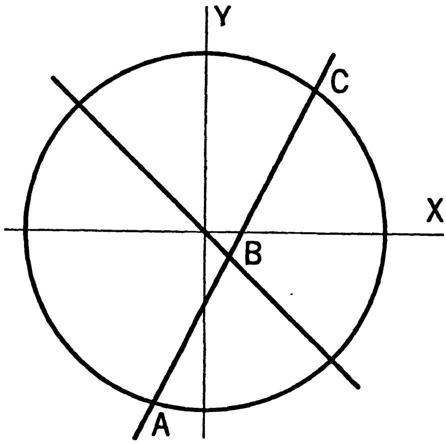


FIG. 17.

14. Find the lengths of the chords joining the points of intersection of $4x^2 + y^2 = 36$ with: (a) $x = 1$; (b) $x = -2$; (c) $y = 1$; (d) $y = -1$.

15. Find the points of intersection of the graphs of

$$(x^2 - y^2)(x^2 - xy - 6y^2) = 0 \text{ and } 2x^2 + 3y^2 = 20.$$

For each of the pairs of equations numbered 16 to 19, sketch the graphs of the equations and shade the areas bounded by them:

16. $y = x^3$, $y = 4x$.

17. $x^2 - y^2 = 1$, $x = \sqrt{4 - y^2}$.

18. $x(y^2 - 4x) = 0$, $y^2 - 4 = 0$.

19. $x^2 = y^3$, $y = \sqrt{2 - x^2}$.

20. What is the greatest possible number of points of intersection of the graphs of $ax^2 + bxy + cy^2 + dx + ey + f = 0$ and $y = mx + n$? Give a reason for your answer.

21. Find the area of the trapezoid having as vertices the points of intersection of $x^2 - y^2 = 3$ and $y^2 = 5 - 2x$.

13. Asymptotes parallel to the axes

Consider the curve

$$y = 1/(x - 2). \tag{9}$$

As x approaches 2 indefinitely close, the corresponding values of y increase without bound. If, for example, x takes on the values

$2 + 1/100$, $2 + 1/1000$, $2 + 1/10^k$, y takes on the values 100, 1000, 10^k. Observe that if k is a large positive number, $2 + 1/10^k$ is very close to 2 and $y = 10^k$ is very large; in fact, for every fixed value M, however large, there is a value of k such that

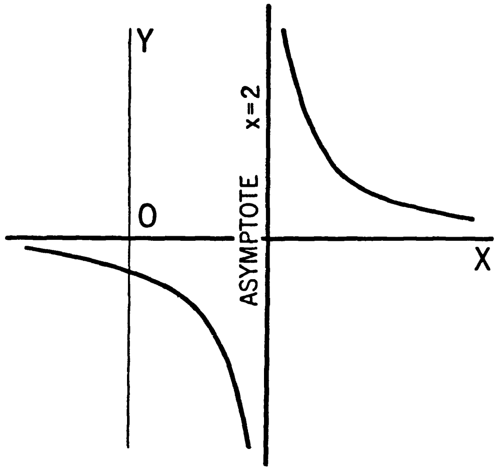


FIG. 18.

$10^k > M$. The line $x = 2$ is called an *asymptote* of (9). Figure 18 shows the graph. Observe that a curve in the neighborhood of an asymptote is indicated by drawing the asymptote and enough of the curve near it to suggest the actual situation.

The solution of (9) for x in terms of y is

$$x = 2 + \frac{1}{y}. \tag{10}$$

From this it appears that, as y approaches zero, x becomes great numerically without bound. Hence, $y = 0$ is an asymptote of (9), and the graph in Figure 18 indicates this fact.

Asymptotes will be considered more in detail in §**61**. Here it is sufficient to note that: *equations of asymptotes parallel to the* y*-axis are obtained by solving the given equation for* y *in terms of* x *and setting the denominator equal to zero; equations of asymptotes parallel to the* x*-axis are obtained by solving the given equation for* x *in terms of* y *and setting the denominator equal to zero.* It is well, in graphing a curve, to discuss it for intercepts, symmetry, extent, and asymptotes and to make a table from which several points may be plotted.

Example. Discuss and sketch the graph of

$$(y - 1)(x^2 - 4) = -12. \tag{a}$$

Solution. By the usual method we find that the

$$x\text{-}intercepts\ are\ \pm 4,\ y\text{-}intercept\ is\ 4, \tag{b}$$

and that *the graph is symmetric to the* y*-axis*. The solutions of (a) for y and for x are

$$y = \frac{x^2 - 16}{x^2 - 4}, \tag{c}$$

$$x = \pm 2\sqrt{\frac{y - 4}{y - 1}}. \tag{d}$$

From (d) we deduce that *there is no curve in the interval for* y *between* 1 *and* 4. From (c) and (d) we find that *the equations of the asymptotes are*

$$x = \pm 2,\ y = 1.$$

Also, using (c), we get the following table of values:

x	0	1	3	4	5
y	4	5	-1.4	0	.43

Figure 19 shows the required graph.

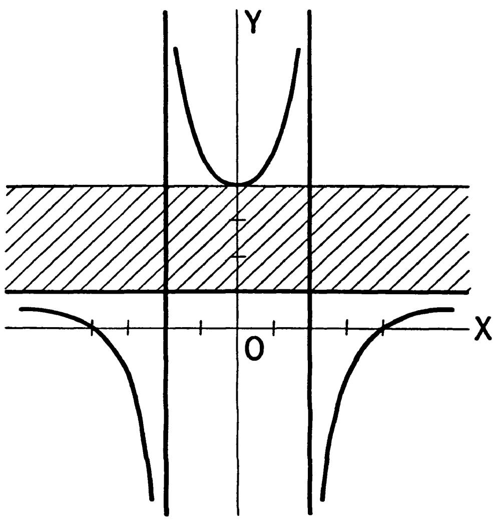


FIG. 19.

Exercises

Discuss and sketch the graph of each of the equations numbered 1 to 18.

1. $xy = 3$.
2. $xy = -3$.
3. $x(y - 1) = 2$.
4. $(x - 2)(y + 2) = 4$.
5. $x^2y = 16$.
6. $(x - 3)^2(y + 2) = 18$.
7. $(y - 3)^2(x + 2) = 18$.
8. $x^2y^2 = 16$.
9. $xy - 2x + 3y - 6 = 0$.
10. $xy + x - 2y + 2 = 0$.
11. $x^2y - x^2 - 4y - 8 = 0$.
12. $y^2x - y^2 - 4x - 8 = 0$.
13. $xy^2 - x = 8$.
14. $y(1 + x^2) = 8$.
15. $(x^2 - 4)y = 2x$.
16. $(x^2 + 16)y^2 = x^2 - 9$.

★17. $y = (x^2 - 4)/(x - 4)$.

★18. $y(x - 2)^2 = x - 4$.

14. Locus of a point

The assemblage of positions that a point may occupy while satisfying some given condition or conditions is called the **locus of the point.** The graph of an equation is also called **the locus of the equation.** At this time only a few very simple loci will be considered. Their equations will be found by expressing the given conditions in terms of x and y.* For example, the equation of the locus of a point whose ordinate is one less than twice its abscissa is $y = 2x - 1$.

Exercises

Write the equation of the loci of points in the xy-plane subject to the conditions numbered 1 to 7.

1. The ordinate is 1 greater than twice the abscissa.
2. The abscissa is one-half the square of the ordinate.
3. The sum of the squares of the coordinates is 25.
4. The square of the ordinate decreased by the square of the abscissa is the sum of the coordinates.
5. The distance from the x-axis is 2.
6. The distance from the y-axis is 7.

* To show that the graph of a derived equation is the locus of a point defined by given conditions, we must demonstrate two of the following three statements: every point on the locus satisfies the equation; every point satisfying the equation lies on the locus; every point not on the locus does not satisfy the equation. In simple situations we are generally content to express given conditions as an equation. When a result is in doubt, a careful investigation should be made.

7. The distance from the x-axis is equal (numerically) to the distance from the y-axis.

8. A point moves in the first quadrant in such a way that the sum of its abscissa and ordinate is 10. Find a point on the locus such that its abscissa is twice its ordinate.

9. What is the distance from point (2,3) to the locus of points having -6 as abscissa?

10. How far apart are the lines (a) and (b) if (a) is the locus of points having ordinate -2 and (b) is the locus of points having ordinate 10?

11. Find all points satisfying the two conditions: the sum of ordinate and abscissa is 10; the square of the ordinate is 4 greater than twice the square of the abscissa.

12. (a) Observing that the distance of (x,y) from the origin (0,0) is $\sqrt{x^2 + y^2}$, find the equation of a circle with center (0,0) and radius 5. (b) Find two points distant 5 from the origin and having an abscissa less by 5 than three times the ordinate.

13. Find the equation of the locus of points equidistant from lines $x = a$ and $x = b$.

CHAPTER III

Fundamental Formulas

15. Distance

Two concepts of basic importance are those of distance and angle. Formulas relating to these will be developed and applied in this chapter.

From each of two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ (see Figure 1), drop perpendiculars to the x-axis meeting it in points A and B

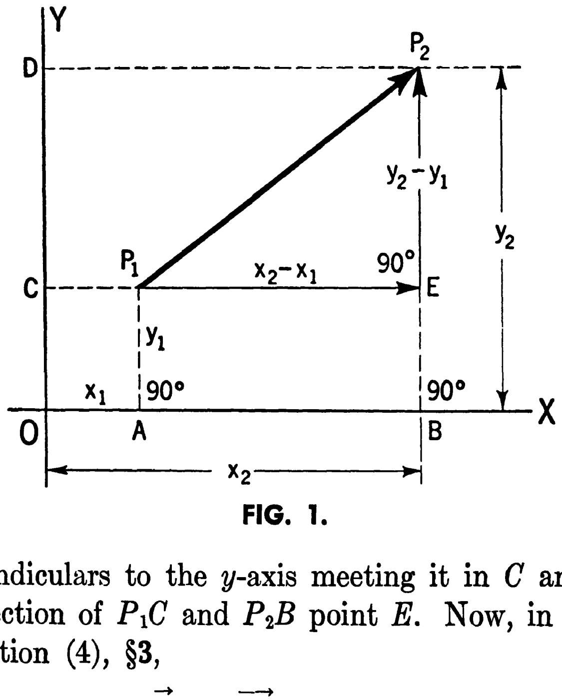


FIG. 1.

and perpendiculars to the y-axis meeting it in C and D. Call the intersection of P_1C and P_2B point E. Now, in accordance with Equation (4), §3,

$$\overrightarrow{AB}(=\overrightarrow{P_1E}) = x_2 - x_1, \tag{1}$$

where x_2 and x_1 are considered in the equation as vectors $\overrightarrow{OB}$ and $\overrightarrow{OA}$. This equation shows that $\overrightarrow{AB}$, $=\overrightarrow{P_1E}$, is equal to $|x_2 - x_1|$ in magnitude and is directed rightward or leftward according as $x_2 - x_1$ is a positive or a negative number.

From this point on, *a number indicating a distance in a certain direction will be called a* **directed distance.** For example, in

Figure 2(a), $x_2 - x_1$ has the magnitude of $\overrightarrow{AB}$ and is negative, corresponding to the fact that $\overrightarrow{AB}$ is directed leftward. In

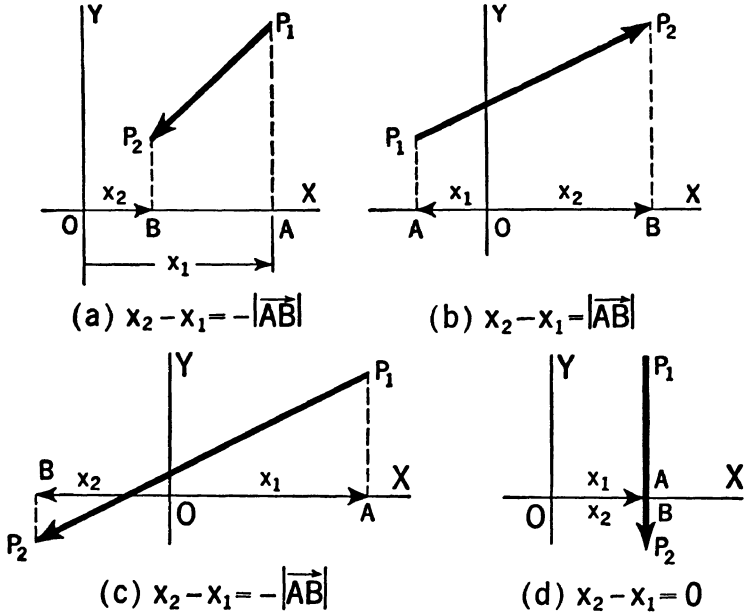


FIG. 2.

Figure 2(b), x_2 is positive and x_1 is negative. Hence, if $x_2 = b$ and $x_1 = -a$, then

$$x_2 - x_1 = b - (-a) = b + a,$$

a positive distance equal in magnitude to $\overrightarrow{AB}$ and, like $\overrightarrow{AB}$, directed rightward. If, in Figure 2(c), $x_1 = a$, a positive number, and $x_2 = -b$, a negative number, then

$$x_2 - x_1 = -b - a,$$

which has the magnitude $|\overrightarrow{AB}|$ and a negative sign indicating the leftward direction of $\overrightarrow{AB}$. In Figure 2(d), $x_2 = x_1$ and $x_2 - x_1 = 0$.

Applying an argument like that leading to Equation (1), we obtain (see Figure 1),

$$\overrightarrow{CD}(=\overrightarrow{EP_2}) = y_2 - y_1, \tag{2}$$

where y_2 and y_1 are considered as vectors in the equation and $y_2 - y_1$ is directed upward or downward according as its value is positive or negative.

From Figure 1 and the Pythagorean theorem,

$$\text{Distance } P_1P_2 = |\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \quad (3)$$

Observe that the distance given by (3) is undirected and is therefore always positive.

Example 1. Show that $A(1,2)$, $B(4,6)$, $C(9,-6)$, and $D(6,-10)$ are the vertices of a parallelogram.

Solution. Substituting 1 for x_1, 2 for y_1, 4 for x_2, and 6 for y_2 in (3), we get

$$AB = \sqrt{(4-1)^2 + (6-2)^2} = \sqrt{9+16} = 5. \quad \text{(a)}$$

Similarly,

$$\begin{aligned} CD &= \sqrt{(6-9)^2 + (-10+6)^2} = 5, \\ BC &= \sqrt{(9-4)^2 + (-6-6)^2} = 13, \\ AD &= \sqrt{(6-1)^2 + (-10-2)^2} = 13. \end{aligned} \quad \text{(b)}$$

Since the opposite sides of the quadrilateral $ABCD$ (see Figure 3) are equal, the figure is a parallelogram.

Example 2. Prove that if the diagonals of a parallelogram are equal, the figure is a rectangle.

Solution. The first step in a proof of this kind is to express the vertices by coordinates so general in nature that any possible figure of the kind under consideration is represented.* Figure 4

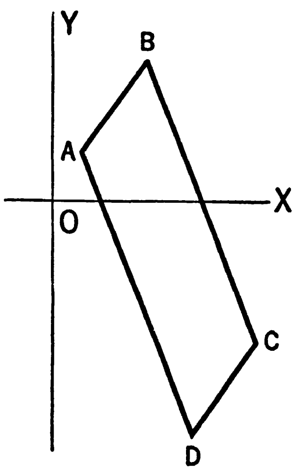


FIG. 3.

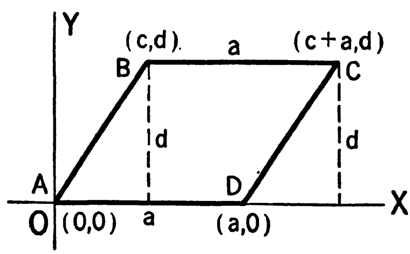


FIG. 4.

represents any parallelogram with the origin taken at one vertex and the x-axis along one side.† Observe that with the coordinates

* The selection of particular numbers for the coordinates of the vertices could lead only to a verification of facts relating to the special figure selected.

† A careful choice of axes will often materially reduce the subsequent work and will not detract from the generality of the proof.

assigned to the vertices in Figure 4, the opposite sides must be equal, and the figure must be a parallelogram if neither a nor d is zero.

Since the diagonals AC and DB are given equal, we have

$$(c + a)^2 + d^2 = (c - a)^2 + d^2$$

or, simplified,

$$4ac = 0.$$

Since by hypothesis $a \neq 0$, c must be zero. But when $c = 0$, Figure $ABCD$ of Figure 4 is a rectangle.

Exercises

1. Find the projections, $x_2 - x_1$ and $y_2 - y_1$ of the segment P_1P_2 on the x-axis and on the y-axis, given:

(a) $P_1(2,3)$, $P_2(3,6)$.
(b) $P_1(5,5)$, $P_2(2,1)$.
(c) $P_1(-3,2)$, $P_2(4,-3)$.
(d) $P_1(-5,-4)$, $P_2(-7,10)$.

2. Find the magnitude of $\overrightarrow{P_1P_2}$, given:

(a) $P_1(1,2)$, $P_2(4, 6)$.
(b) $P_1(-4,3)$, $P_2(2,-5)$.
(c) $P_1(-1, 6)$, $P_2(2,6)$.
(d) $P_1(-2,-3)$, $P_2(5,-4)$.

3. Plot the points (0,0), (5,0), (5,3), and (0,3), observe that they are the vertices of a rectangle, and then show that its diagonals are equal.

4. Observe that Figure 5 represents any rectangle. Prove that the diagonals of a rectangle are equal.

5. Show that $ABCD$ is a parallelogram if its vertices are:

(a) $A(4,5)$, $B(3,3)$, $C(0,6)$, $D(1, 8)$.
(b) $A(2,3)$, $B(4,6)$, $C(7,1)$, $D(5,-2)$.

6. Show that the triangle with vertices $A(-2,7)$, $B(6,5)$, and $C(0,-1)$ is isosceles.

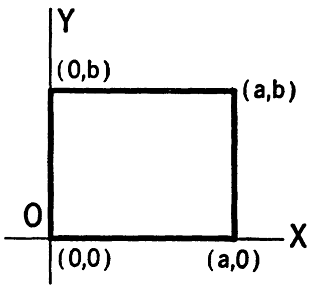


FIG. 5.

In the exercises numbered 7 to 12, obtain the answers from sketches by inspection.

7. Find x and y if the vertices of a rectangle are:

(a) (0,0), (6,0), (x,y),(0,4).
(b) (0,0), $(a,0)$, (x,y),$(0,b)$.

8. Find x and y if the vertices A,B,C, and D of a parallelogram with diagonal BD are:

(a) $A(0,0)$, $B(6,0)$, $C(x,y)$, $D(3,5)$.
(b) $A(0,0)$, $B(a,0)$, $C(x,y)$, $D(c,d)$.

9. In a triangle ABC, $AC = BC$ and the altitude from vertex C has length h. Find the coordinates of C if A and B are the points:

(a) $A(-a,0)$, $B(a,0)$. (b) $A(0,0)$, $B(b,0)$.

10. Two vertices of an equilateral triangle are the points $(0,0)$ and $(a,0)$. Find the third vertex.

11. One vertex of a square is at the origin and a diagonal lies along the positive x-axis. Find the coordinates of the other vertices if: (a) the length of a diagonal is 6; (b) the length of a side is s.

12. The longer base of an isosceles trapezoid extends from $(0,0)$ to $(8,0)$. The lengths of the shorter base and the altitude are 4 units and 6 units, respectively. Find the coordinates of the ends of the shorter base.

13. Show that a line joining the midpoints of the sides AB and BC of triangle ABC is equal to one half of the third side if the points A,B, and C are: (a) $A(0,5)$, $B(16,11)$, $C(12,0)$; (b) $A(6,7)$, $B(2,-3)$, $C(-2,1)$.

14. Show that the triangle with vertices $A(1,0)$, $B(3,-4)$, and $C(-3,-2)$ is a right triangle. Recall that a triangle is a right triangle if the sum of the squares of its two shorter sides is equal to the square of its longest side.

15. Show that ABC is a right triangle if its vertices are:

(a) $A(3,5)$, $B(3,-1)$, $C(7,-1)$. (b) $A(1,2)$, $B(3,1)$, $C(-4,-8)$.

16. Show that ABC is an isosceles right triangle if its vertices are: (a) $A(3,3)$, $B(11,-3)$, $C(9,11)$. (b) $A(3,4)$, $B(6,-3)$, $C(-4,1)$.

17. Show that points $A(0,3)$, $B(8,-3)$, and $C(-4,6)$ lie in a straight line. Points P,Q, and R are in a straight line provided that Q lies between P and R and $PQ + QR = PR$.

18. Prove that points $A(1,-4)$, $B(3,0)$, and $C(-2,-10)$ are in a straight line.

19. Express in the form of an equation in x and y the condition that point (x,y) be: (a) 5 units from $(3,-2)$; (b) 4 units from the origin; (c) equidistant from $(0,1)$ and $(2,3)$; (d) equidistant from $(1,-2)$ and $(5,4)$; (e) twice as far from $(0,0)$ as from $(3,5)$.

20. Express that a point (m,n) lies on the line $x + y = 5$ and that it is 4 units from $(2,-1)$. Solve the results for m and n, thus finding two points on $x + y = 5$ which are distant 4 units from $(2,-1)$.

21. Find two points on line $2x - y = 2$ each 5 units from the origin. First read Exercise 20.

22. Show that the following points are vertices of a square: $(1,2)$, $(9,8)$, $(15,0)$, $(7,-6)$.

23. Show that $(1,2)$ is the center of a circle through $(2,6)$, $(5,1)$, and $(0,-2)$, and also that $(-4,3)$ is the center of a circle through $(-2,6)$, $(-7,1)$, and $(-6,6)$.

Find the directed distances in Exercises 24 to 28. In this connection, think of up as +, down as −, right as +, and left as −:

24. From the y-axis to (7,8).

25. From the x-axis to (6,4).

26. From a line parallel to the y-axis and 2 units left of it to $(-8,-8)$.

27. From a line parallel to the x-axis and 2 units above it to $(4,-8)$.

28. From a line parallel to the y-axis and 4 units to the right of it to $(-6,-10)$.

29. Find x if the distance from (1,2) to $(x,-4)$ is 10.

30. Find y if the distance from (3,4) to $(-2,y)$ is 13.

From each pair of equations numbered 31 to 33, find the length of the part of the line forming a chord of the curve:

31. $x^2 + y^2 = 125,\ 2x - y = 20.$

32. $4x^2 + y^2 = 200,\ x + y = 15.$

33. $x^2 - y^2 = 7,\ 2x - y = 5.$

34. Show that the area A of a triangle with vertices $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, and $P_3(x_3,y_3)$ is given by

$$A = \pm \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.^*$$

Hint. From Figure 6 find the areas of the trapezoids AP_3, CP_3, and AP_2, and combine them to get the area of triangle $P_1P_2P_3$. Then expand the determinant.

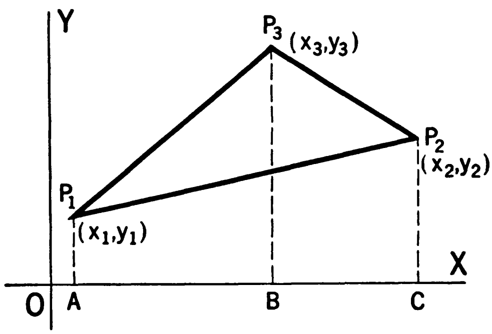


FIG. 6.

35. Using the formula in Exercise 34, find the area of the triangle having as vertices: (a) (2,3), $(1,-2)$, and (3,1); (b) (0,0), $(2,-6)$, and (4,3).

36. Prove that the sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of its diagonals.

* The sign + or − that will give a positive number for A should be used.

16. Angle. Inclination. Slope

*The **angle** which a line* l_2 *makes with a line* l_1 *is the least angle through which* l_1 *must be rotated counterclockwise* to occupy a position parallel† to* l_2. Observe that this angle is always less than 180°. *The **inclination** of a line is the angle the line makes with the* x-*axis.* It is always positive and less than 180°.

Thus, in Figure 7, line AB makes an angle of 60° with line CD,

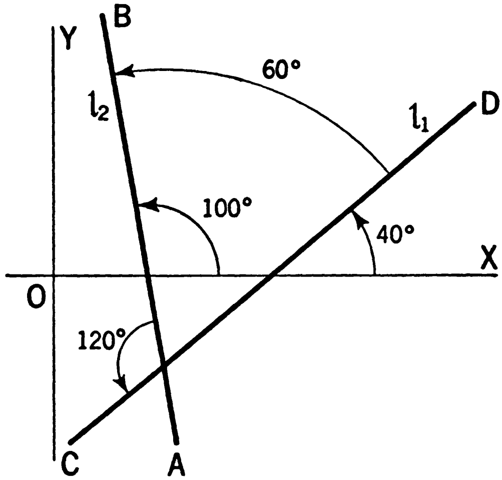


FIG. 7.

and line CD makes an angle of 120° with line AB. The inclination of CD is 40° and the inclination of AB is 100°.

*The **slope** of a line is the tangent of its inclination.* Let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ be two points on a line of inclination θ (see Figure 8). Then, in accordance with the definition of the tangent of an angle, we have

$$\text{Slope } P_1P_2 = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1},\ x_2 - x_1 \neq 0. \tag{4}$$

Observe that the slope may be written $(y_1 - y_2)/(x_1 - x_2)$, since numerator and denominator may be multiplied by -1

* Imagine a clock with its face upward lying on the book. Its hands rotate and generate angles. A line turning in a sense opposite to that of the hands of the clock is said to turn counterclockwise. In general, angles generated by counterclockwise rotation are considered positive and angles generated by clockwise rotation are considered negative.

† Coincident lines or collinear segments are here regarded as instances of parallel lines or segments.

without changing the value of the fraction. Also note that when the inclination θ is less than $90°$, the slope is zero or positive; when θ is greater than $90°$, the slope is negative; and when $\theta = 90°$, there is no slope.

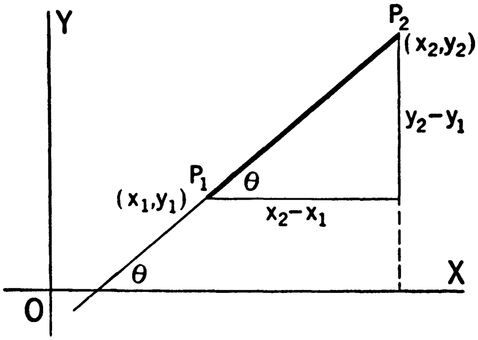


FIG. 8.

17. The angle that one line makes with another

The angle ϕ which a line AB of inclination θ_2 makes with a line CD of inclination θ_1 is given by either

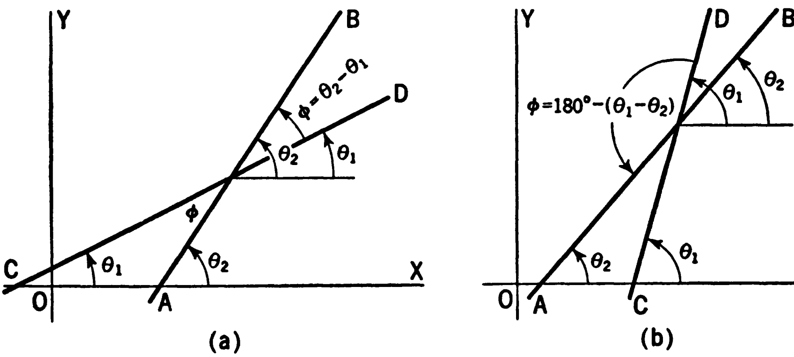
$$\phi = \theta_2 - \theta_1,$$

or

$$\phi = 180° - (\theta_1 - \theta_2) = 180° + (\theta_2 - \theta_1), \tag{5}$$

as indicated in Figure 9. Figure 9(a) represents the case where $\theta_2 \geqq \theta_1$, and Figure 9(b) the case where $\theta_1 > \theta_2$. In either case,

$$\tan\phi = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2} = \frac{m_2 - m_1}{1 + m_1 m_2}, \quad 1 + m_1 m_2 \neq 0, \tag{6}$$



Angle Which AB Makes with CD = $\theta_2 - \theta_1$.

Angle Which AB Makes with CD = $180° - (\theta_1 - \theta_2)$.

FIG. 9.

where $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$ are the respective slopes of the lines involved. If $1 + m_1m_2 = 0$, $\tan\phi$ does not exist and, as will appear below, the lines are mutually perpendicular. If the value of $\tan\phi$ given by (6) is positive, the angle specified by ϕ is between 0° and 90°; if $\tan\phi$ is negative, ϕ is between 90° and 180°. *Observe that the angle ϕ is measured from the line of slope* m_1.

If two lines have the same slope, that is, if

$$m_1 = m_2, \tag{7}$$

they are parallel; for their inclinations are equal, since $\tan\theta_1 = \tan\theta_2$ and θ_1 and θ_2 are both less than 180°. Conversely, if the lines are parallel, they have the same inclinations and (7) holds true unless their inclinations are 90°.

When the lines are perpendicular to each other, $\phi = 90°$, and, therefore, from (5), either

$$\theta_2 = 90° + \theta_1, \text{ or } \theta_2 = \theta_1 - 90°. \tag{8}$$

In either case, unless $\theta_2 = 0$ or $\theta_1 = 0$, $\tan\theta_2 = -\cot\theta_1 = -1/\tan\theta_1$, and

$$m_2 = -\frac{1}{m_1}. \tag{9}$$

In words, *if two lines are mutually perpendicular, the slope of one is the negative reciprocal of the slope of the other.* Conversely, if (9) holds true, then

$$\tan\theta_2 = -\cot\theta_1 = \tan(90° + \theta_1 \pm k\,180°), \tag{10}$$

where k is a positive integer, and, therefore,

$$\theta_2 = 90° + \theta_1 \pm k\,180°. \tag{11}$$

From (11) and the fact that θ_1 and θ_2 are both positive angles less than 180°, one or the other of equations (8) holds, and the lines are perpendicular to each other.

In summary, *equation* (6) *specifies the angle which a line of slope* m_2 *makes with a line of slope* m_1. *If* $m_1 - m_2 = 0$, *the lines are parallel; if* $1 + m_1m_2 = 0$, *the lines are perpendicular to each other. If either* m_1 *or* m_2 *does not exist, one of the lines is parallel to the* y-*axis.*

Example. Find the angles A and B of the triangle having as vertices $A(4,3)$, $B(9,-4)$, and $C(-3,0)$.

Solution. Figure 10 shows the triangle. To get angle B, take for m_1 the slope of BA, since examination of the figure shows that angle B is measured from BA, and for m_2, the slope of BC. Hence, using (4), obtain

$$m_1 = \frac{3-(-4)}{4-9} = -\frac{7}{5}\,; m_2 = \frac{-4-0}{9+3} = -\frac{1}{3}.$$

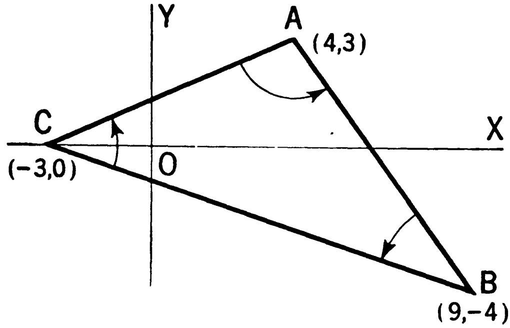


FIG. 10.

Substituting these values in (6), obtain

$$\tan B = \frac{-\frac{1}{3}-(-\frac{7}{5})}{1+(-\frac{1}{3})(-\frac{7}{5})} = \frac{8}{11} = 0.727.$$

Hence, from the table on page 260, $B =$ **36.0°**. To get angle A, take m_1 as the slope of AC and m_2 as the slope of AB to obtain $m_1 = \frac{3}{7}$, $m_2 = -\frac{7}{5}$. Using these values in (6), obtain

$$\tan A = \frac{-\frac{7}{5}-\frac{3}{7}}{1+(-\frac{7}{5})(\frac{3}{7})} = -\frac{32}{7} = -4.57.$$

From the table on page 261, $\tan^{-1} 4.57 = 77.7°$. Hence, angle $A = 180° - 77.7° =$ **102.3°**.

Exercises

1. Find the slope and inclination of the line connecting the points: (a) (1,2), (5,6). (b) (2,−5), (7,−2). (c) $(-1, \sqrt{3})$, $(2, -2\sqrt{3})$. (d) (−7,5), (7,33). (e) (5,6), (10,6).

2. Find the slope of the x-axis and of the y-axis.

3. Find the slope of the line perpendicular to the line through: (a) (2,5) and (4,−6). (b) (5,−2) and (6,3). (c) (10,6) and (10,10). (d) (10,7) and (−4,7).

4. Find the slopes of the lines having inclinations of 40°, 75°, 120°, 0°, 90°.

5. Find the slopes of the sides and of the altitudes of triangle ABC where the vertices are $A(1,0)$, $B(-3,4)$, $C(4,-4)$.

6. State which of the lines AB, AC, AD, BC, BD, and CD are parallel and which perpendicular, where the points referred to are $A(2,3)$, $B(-3,-7)$, $C(5,-3)$, and $D(-6,-1)$.

7. Find angle ABC if the points which define the angle are: (a) $A(2,3)$, $B(5,7)$, $C(4,-5)$; (b) $A(5,4)$, $B(1,2)$, and $C(4,-4)$.

8. Find the angle that the line connecting $(-2,3)$ to $(4,-6)$ makes with a line of inclination $\tan^{-1}(\frac{1}{2})$.

9. Show that points $(2,5)$, $(8,-1)$, and $(-2,1)$ are vertices of a right triangle, and find its angles.

10. Find the angles of the triangle having vertices $(-2,-3)$, $(3,4)$, and $(-5,5)$.

11. Find the angles of the parallelogram having vertices:
(a) $(-4,3)$, $(0,-3)$, $(7,-1)$, $(3,5)$; (b) $(0,3)$, $(4,-1)$, $(-2,-1)$, $(2,-5)$.

12. Prove that points $(3,6)$, $(8,-4)$, and $(-6,24)$ lie on a straight line.

13. Determine which of the points $(3,-10)$, $(-1,3)$, $(-3,11)$, and $(5,-18)$ are on the line through $(1,-3)$ and $(-1,4)$.

14. Write the condition that a line through (x,y) and $(5,6)$ be perpendicular to a line through $(4,5)$ and $(-2,-5)$.

15. Write the condition that a line through (x,y) and $(2,3)$ have slope 2.

16. Write the condition that a line joining (x,y) to $(1,-2)$ be parallel to the line through $(5,6)$ and $(7,-3)$.

17. Write the condition that the angle which a line through $(2,3)$ and (x,y) makes with the line through $(2,3)$ and $(5,-6)$ be 45°.

18. Find the angle which a line of slope -3 makes with the line along which ordinate equals abscissa.

19. Find the slope of a line making an angle of 45° with the line through $(-2,-2)$ and $(3,6)$.

20. The tangent of the angle made by a line of slope $\frac{2}{3}$ with a second line is 1. Find the slope of the second line.

21. A line of slope 3 passes through $(0,10)$. If a point on the line has abscissa -3, find its ordinate.

22. Line AB is perpendicular to line CD, where the points referred to are $A(5,6)$, $B(4,-2)$, $C(3,-1)$, and $D(a,12)$. Find a.

23. Prove that the line AB bisects angle DAC, where the points defining the line and angle are $A(2,-3)$, $B(9,-2)$, $C(8,5)$, and $D(6,-6)$.

24. Prove by slopes that the diagonals of a square are perpendicular to each other.

25. Prove that the diagonals of a rhombus are perpendicular to each other. (See Figure 11).

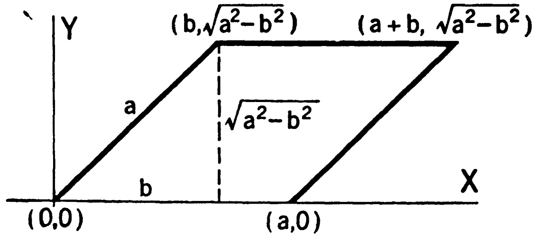


FIG. 11.

★26. Find the slope of a line bisecting the angle which a line of slope -2 makes with a line of slope 3.

18. An angle which one directed line makes with another

For many purposes in mathematics and its applications, it is convenient to speak of the angles defined by *directed* lines. *An angle which a directed line* l_2 *makes with a directed line* l_1 *is an angle through which* l_1 *must be turned to have the same directior as* l_2. Thus, in Figure 12(a), vector $\overrightarrow{AB}$ makes an angle of 60° or of $-300°$ with $\overrightarrow{AC}$. Figure 12(b) shows the angles $-60°$, 300°, and

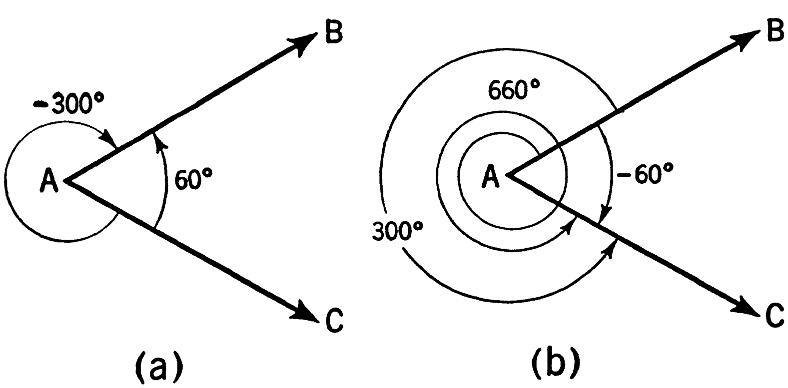


FIG. 12.

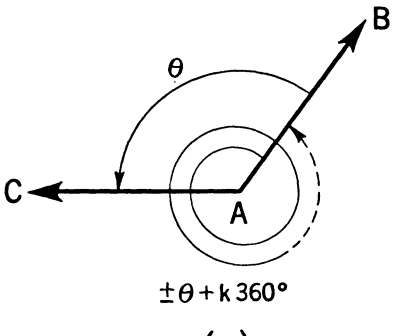


FIG. 12 (Cont.).

$300° + 360° = 660°$ which $\overrightarrow{AC}$ makes with $\overrightarrow{AB}$. Figure 12(c) represents the general case; a little inspection shows that any angle which either vector $\overrightarrow{AB}$ or vector $\overrightarrow{AC}$ makes with the other will be included among the angles

$$\pm\theta + k\,360°, \tag{12}$$

where k is an integer or zero.

It is interesting to observe that the cosines of all the angles (12) are equal, since

$$\cos\,(\pm\theta + k\,360°) = \cos\,\theta. \tag{13}$$

Exercises

Find the non-negative angle less than 360° which each of the vectors specified in Exercises 1 to 6 makes with the vector from the origin to: (a) (1,0); (b) (0,1):

1. From (0,0) to (0,1).

2. From (0,0) to $(-1,0)$.

3. From (0,0) to $(0,-1)$.

4. From (0,0) to (1,1).

5. From (0,0) to $(-2,2\sqrt{3})$.

6. From (0,0) to $(-2\sqrt{3},-2)$.

Find the negative angle in the range from $-360°$ to 0° that each of the vectors specified in Exercises 7 to 10 makes with the vector from the origin to $(0,-1)$:

7. From (0,1) to (1,0).

8. From (0,1) to $(-1,0)$.

9. From (1,1) to (2,2).

10. From $(2,2\sqrt{3})$ to $(-2,-2\sqrt{3})$.

Find the angle in the range 360° to 719° which each of the vectors specified in Exercises 11 to 14 makes with the vector from (1,1) to $(-1,-1)$:

11. From $(-1,-1)$ to (1,1).

12. From (0,0) to $(1,-3)$.

13. From (1,1) to (2,3).

14. From $(3,-1)$ to $(-3,-6)$.

15. Find a positive angle and a negative angle, each less in magnitude than 360°, which $\overrightarrow{AB}$ makes with $\overrightarrow{AC}$, where $A,B,$ and C are the respective points (1,2), $(6,-3)$, and $(-4,-3)$.

16. A wheel turned through five revolutions and a part of a sixth one, clockwise. Through what angle did it turn if a spoke in the wheel directed from center to circumference had initial inclination $\tan^{-1}\frac{1}{2}$ to an x-axis fixed in its plane of motion and a terminal inclination $\tan^{-1}(-\frac{1}{2})$ to the same x-axis?

19. Point of division

Let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$, where $x_1 \neq x_2$ and $y_1 \neq y_2$, be two points in the xy-plane and $P(x,y)$ another point on line P_1P_2 (see Figure 13). Then,

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}, \tag{14}$$

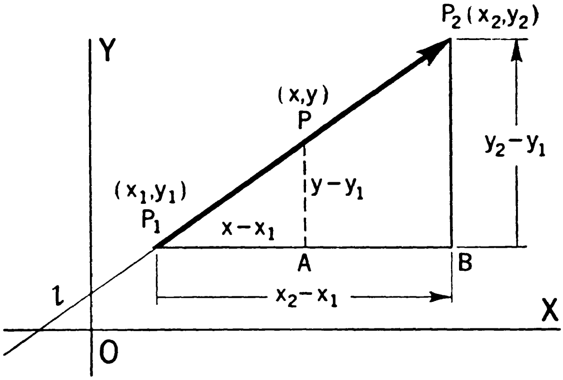


FIG. 13.

since each member of the equation is the slope of the line P_1P_2. Multiplying the members of (14) by $(x - x_1)/(y_2 - y_1)$ and equating the equal ratios thus obtained to r, we have, after slight simplification,

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} = r; \tag{15}$$

or, solving for x and y,

$$x = x_1 + r(x_2 - x_1), y = y_1 + r(y_2 - y_1). \tag{16}$$

By interpreting Equation (15) in connection with the similar triangles P_1AP and P_1BP_2 of Figure 13, one deduces that r is the ratio of P_1P to P_1P_2. If r is taken as $\frac{1}{2}$ in (15) or (16), we get as *the coordinates of the midpoint of* P_1P_2:

$$x = \frac{x_1 + x_2}{2}, y = \frac{y_1 + y_2}{2}. \tag{17}$$

Similarly, we could obtain the trisection points of P_1P_2 by replacing r in (16) by $\frac{1}{3}$ and $\frac{2}{3}$ in succession. Using $r = 2$ in (16), we get a point P distant $2\,|\overrightarrow{P_1P_2}|$ from P_1 along the line through P_1 and P_2 and with $\overrightarrow{P_1P} = 2\overrightarrow{P_1P_2}$; taking $r = -2$ in (16), we get a point P distant $2\,|\overrightarrow{P_1P_2}|$ from P_1 but with $\overrightarrow{P_1P} = -2\overrightarrow{P_1P_2}$.

Example. Find the point of intersection of the medians of a triangle (see Figure 14) with vertices $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, and $P_3(x_3,y_3)$.

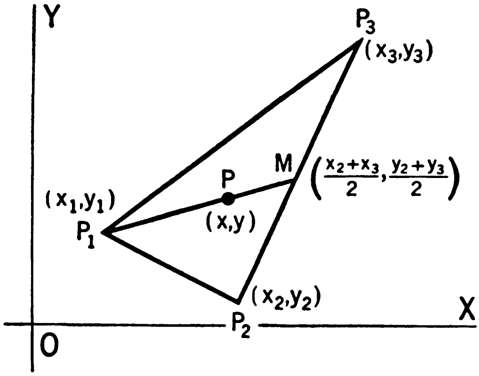


FIG. 14.

Solution. By formulas (17), the coordinates of M, the midpoint of P_2P_3, are

$$x_M = \frac{x_2 + x_3}{2}, \; y_M = \frac{y_2 + y_3}{2}. \tag{a}$$

Now the required point is on P_1M two-thirds of the way from P_1 to M. Hence, using (16), with $r = \frac{2}{3}$ and x_M and y_M from (a) taking the place of x_2 and y_2, we get for the required point P:

$$x = x_1 + \frac{2}{3}\left(\frac{x_2 + x_3}{2} - x_1\right) = \frac{x_1 + x_2 + x_3}{3},$$

$$y = y_1 + \frac{2}{3}\left(\frac{y_2 + y_3}{2} - y_1\right) = \frac{y_1 + y_2 + y_3}{3}.$$

Exercises

1. Find the midpoints of the sides of the triangle having vertices $(-3,2)$, $(3,6)$, and $(-4,8)$.

2. Using Formula (16), find the two points between $A(5,6)$ and $B(11,-12)$ dividing the segment AB into three equal parts.

3. Two similar triangles ABC ($A(0,0)$, $B(6,0)$, $C(0,9)$) and $AB'C'$ lie on the same side of AB and corresponding sides have the same directions. Find points B' and C' if the ratio of AB' to AB is: (a) $\frac{1}{2}$; (b) $\frac{1}{3}$; (c) 2.

4. In Exercise 3 replace $C(0,9)$ by $C(12,9)$ and solve the resulting problem.

5. Apply formula (16) for $P_1(1,2)$, $P_2(-5,8)$ to find the two points twice as far from P_1 as from P_2.

6. In formulas (16) replace r by $r_1/(r_1 + r_2)$ to find formulas for the coordinates of the point P which divides the segment P_1P_2 in the ratio $r_1 : r_2$:

$$x = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \qquad y = \frac{r_1y_2 + r_2y_1}{r_1 + r_2}.$$

7. Use the formulas in Exercise 6 to find the point dividing P_1P_2 in the ratio r_1 to r_2 where the reference points are: (a) $P_1(2,-4)$, $P_2(6,14)$, and $r_1/r_2 = \frac{1}{3}$. (b) $P_1(-18,0)$, $P_2(0,54)$, $r_1/r_2 = 4$.

8. In Figure 14 find the point two-thirds of the way from P_2 to the midpoint of P_1P_3, and also the point two-thirds of the way from P_3 to the midpoint of P_1P_2.

9. Find the point of intersection of the medians of the triangle ABC where the vertices are: (a) $A(-3,-2)$, $B(4,5)$, $C(2,-9)$; (b) $A(5,-6)$, $B(-1,8)$, $C(4,6)$.

10. Show that the midpoint of the hypotenuse of a right triangle is equidistant from its vertices.

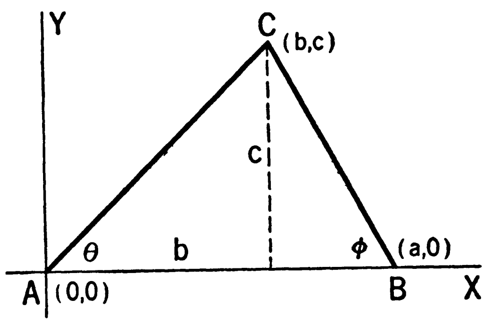


FIG. 15.

11. Show that the line joining the midpoints of the non-parallel sides of a trapezoid is parallel to the parallel sides and equal to one-half the sum of their lengths.

12. Prove that the midpoints of the sides of a quadrilateral are the vertices of a parallelogram.

13. Prove that a triangle is isosceles if two of its medians are equal in length.

14. Figure 15 represents any triangle. Using the figure, prove the law of cosines, Equation 35, page 259, and the law of sines, Equation 34, page 259, from trigonometry.

Hint. Verify that $\overline{BC}^2$ and $\overline{AB}^2 + \overline{AC}^2 - 2\overline{AB}\ \overline{AC} \cos\theta$ are equal.

15. Show that in any quadrilateral the lines joining the midpoints of the opposite sides, and the line joining the midpoints of the diagonals, meet in a point and bisect each other.

20. Locus problems

Since our power of expressing geometric conditions analytically is much greater than it was when we studied §**14**, we can now find equations of more complicated loci. The first step in solving any locus problem is to draw a figure displaying the given data and showing (x,y), a typical point which satisfies the given conditions, chosen generally in the first quadrant. Next the given conditions should be expressed in terms of x and y and the resulting equation simplified.

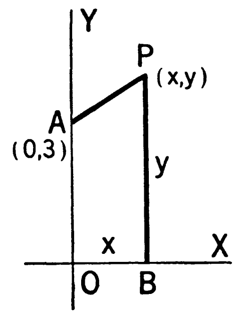


FIG. 16.

Example. Find the locus of a point twice as far from the x-axis as from the point (0,3).

Solution. Figure 16 shows the point (x,y) in a typical position. The distance of (x,y) from A is $\sqrt{(x-0)^2 + (y-3)^2}$ in accordance with distance formula (3), §**15**. The distance of P from the x-axis is seen to be y. Since the two distances y and $2\sqrt{(x-0)^2 + (y-3)^2}$ are numerically equal, their squares are equal, and we have

$$y^2 = [2\sqrt{x^2 + (y-3)^2}]^2 = 4(x^2 + y^2 - 6y + 9),$$

or, simplifying,

$$\mathbf{4x^2 + 3y^2 - 24y + 36 = 0.}$$

Transforming this by the translation

$$x = x',\ y = y' + 4,$$

we obtain

$$4x'^2 + 3y'^2 = 12.$$

Sketching both sets of axes and the curve, we obtain the graph shown in Figure 17.

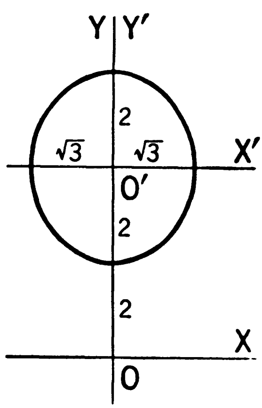


FIG. 17.

Exercises

1. Find the equation of the locus of a point equidistant from (2,3) and (−4, 6).

Find the equations of the loci of points described in Exercises 2 to 9 and sketch the loci:

2. Twice as far from (0,2) as from (0,8).

3. Equidistant from (0,0) and (2,6).

4. Twice as far from (−2,0) as from (−4,0).

5. Equidistant from (4,0) and the y-axis.

6. Equidistant from (0,−2) and the x-axis.

7. Equidistant from (−2,3) and line $x = -4$.

8. Equidistant from (−2,3) and line $y = -4$.

9. Equidistant from (−2,3) and $x = 8$.

10. Find the equation of the locus of a point which moves so that the sum of the squares of its distances from $A(3,0)$ and $B(-3,0)$ is equal to the square of the distance AB.

11. Find the equation of the locus of a point which moves so that the sum of the squares of its distances from (2,−2) and (4,2) is always 28.

12. Replace *sum of the squares* by *difference of the squares* in Exercise 11, and solve the resulting problem. Get two answers.

13. Find the equation of the straight line having slope 3 and passing through (2,−2).

Hint. Figure 18 represents the line. Find the slope of AP, equate it to 3, and simplify.

By using the method of equating two expressions for the slope of a line, find the equations of the lines satisfying the conditions numbered 14 to 21:

14. Through (2,−2) and having slope −2.

15. Through (0,0) and (5,6).

16. Through (5,−5) and (6,−2).

17. Through (0,2) and (5,3).

18. Through (5,0) and (0,7).

19. Through (0,5) and having inclination 45°.

20. Through (−3,0) and having inclination 150°.

21. Through (2,2) and making an angle of 45° with the line connecting (2,2) and (5,−4).

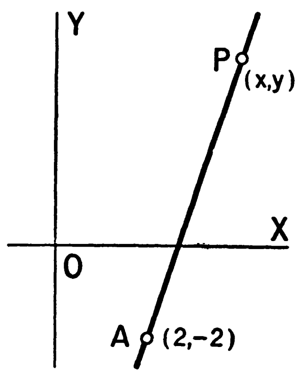


FIG. 18.

To find the equation of a circle, equate the length of its radius to the expression for the distance from any point (x,y) on it to its center. Find the equations of the circles satisfying the conditions mentioned in Exercises 22 to 26:

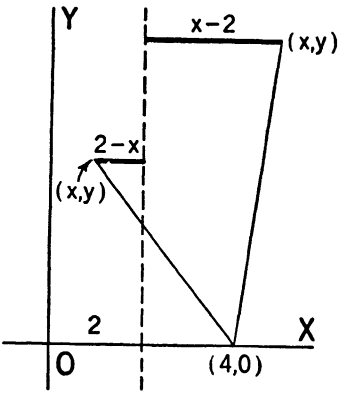


FIG. 19.

22. Center (0,2), radius 2.

23. Center $(a,0)$, radius a.

24. Center (0,3) and passing through (0,0).

25. Center (3,4) and tangent to the x-axis.

26. Having $(3,-2)$ and $(-5,4)$ as ends of a diameter.

To find points satisfying two conditions, find the equations of two loci and solve them simultaneously. Find the points satisfying the conditions mentioned in Exercises 27 to 32:

27. On the x-axis and on the line of slope 3 through (2,10).

28. On the y-axis and on the circle with center (1,4) and radius 5.

29. Equidistant from (1,1) and $(3,-3)$, and also equidistant from $(1,-1)$ and $(2,-2)$.

30. Distant 5 from (1,3) and lying on the x-axis.

31. Twice as far from (3,0) as from the y-axis and also on a line of slope 2 through $(0,-2)$.

32. Distant 20 from the origin and 15 from point (25,0).

33. Find the equation of the locus of point P so situated that the slope of AP is one-third the slope of BP, where A is the origin and B is point (3,5).

★34. Find the equations of the locus of a point 4 units farther from (4,0) than from the line $x = 2$. Sketch the locus.

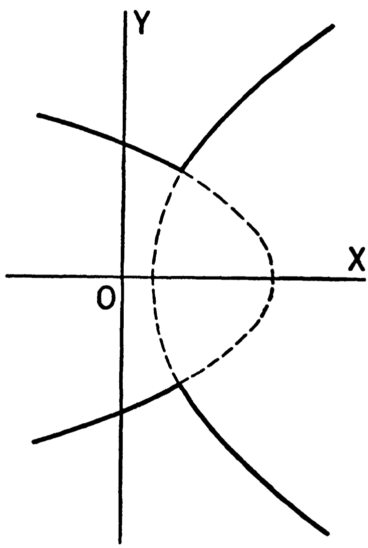


FIG. 20. The dotted parts of the curves are not parts of the locus.

Hint. Essentially, the equation is $\sqrt{(x-4)^2+y^2} = |x-2| + 4$. Hence (see Figure 19),

if $x - 2 > 0$, then $(\sqrt{(x-4)^2+y^2})^2 = (x - 2 + 4)^2$,

if $x - 2 < 0$, then $(\sqrt{(x-4)^2 + y^2})^2 = (-x + 2 + 4)^2$,

and there are two distinct loci. Figure 20 represents the locus.

★**35.** Find the equations of the locus of a point 4 units farther from point $(0,-1)$ than from $y = 1$.

★**36.** Find the equation of the locus of a point 1 unit farther from $(0,-1)$ than from $y = 1$.

★**37.** Find the equations of the locus of a point the sum of whose distances from the point $(0,0)$ and the line $x = -3$ is 6. Sketch the locus.

CHAPTER IV

The Straight Line

21. Equation of a straight line by slopes

A rather general method of finding the equation of a straight line consists in finding two expressions for the slope, one involving point (x,y), *equating them, and simplifying the result.*

Consider, for example, the problem of finding the equation of a line through (2,3) and having 1 as slope. Let (x,y) be any

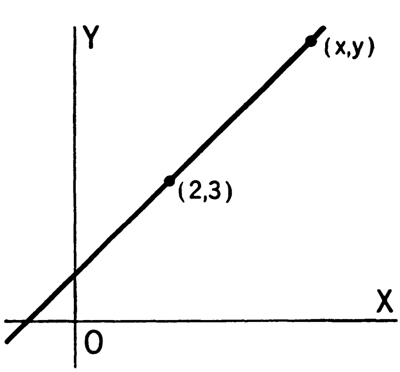


FIG. 1.

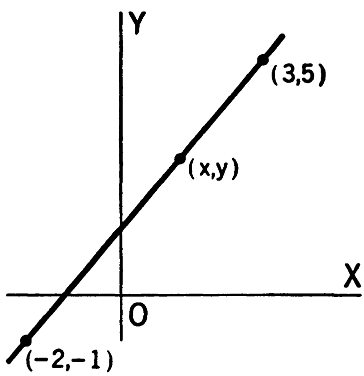


FIG. 2.

point on the line (see Figure 1); then two expressions for the slope are 1 and $(y - 3)/(x - 2)$. Equating these and simplifying, we obtain

$$\frac{y-3}{x-2} = 1, \quad \text{or} \quad x - y + 1 = 0.$$

To find the equation of a line through $(-2,-1)$ and (3,5), let (x,y) be any point on the line (see Figure 2), and find from points (x,y) and $(-2,-1)$ the slope $(y + 1)/(x + 2)$ and from points (3,5) and $(-2,-1)$ the slope $(5 + 1)/(3 + 2)$ or 6/5.

Equating the two values of the slope and simplifying, we obtain

$$\frac{y+1}{x+2} = \frac{6}{5}, \quad \text{or } 6x - 5y + 7 = 0.$$

A slightly more difficult problem is that of finding the equation of a line having y-intercept 4 and cutting at right angles the line through $A(2,-1)$ and $B(4,5)$. To say that the y-intercept is 4 is equivalent to saying that the line passes through (0,4). Figure 3 indicates essential relations. Since the slope of line AB is $(5+1)/(4-2) = 3$, the slope of the required line is the negative reciprocal or $-\frac{1}{3}$. Since the slope is also $(y-4)/(x-0)$, we have

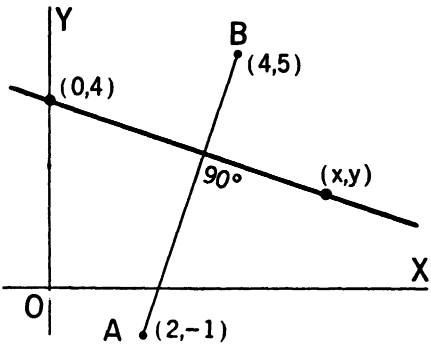


FIG. 3.

$$\frac{y-4}{x} = -\frac{1}{3}, \quad \text{or} \quad y = -\frac{1}{3}x + 4.$$

Exercises

Find the equations of the lines specified in the exercises numbered 1 to 19:

1. Through (2,2) with slope 2.

2. Through (5,6) and (2,−1).

3. Through (2,−1) and (−25,26).

4. Through (3,4) and having y-intercept 3.

5. Through (3,4) and having x-intercept 3.

6. Having x-intercept 5 and y-intercept −4.

7. Through (2,3) and perpendicular to the line through (0,0) and (5,4).

8. Having x-intercept −3 and parallel to the line through (2,5) and (6,3).

9. Through points (x_1,y_1) and (x_2,y_2).

10. Through point (x_1,y_1) and having slope m.

11. Having y-intercept b and slope m.

12. Having x-intercept a and y-intercept b.

13. Having x-intercept a and slope m.

14. Having inclination 30° and x-intercept 5.

15. Having inclination 120° and y-intercept -6.

16. Having inclination θ and y-intercept b.

17. Passing through (4,5) and meeting the line through the origin and (4,5) at right angles.

18. Passing through (4,5) and making an angle of 45° with the line through (0,0) and (4,5).

Hint. Use Equation 6, §**17**, and the ideas suggested by Figure 4.

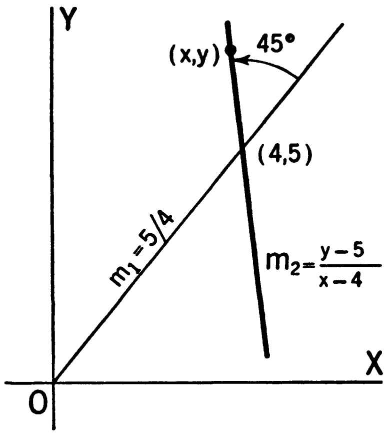


FIG. 4.

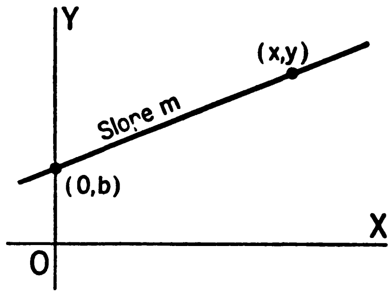


FIG. 5.

★**19.** Passing through $(5,-2)$ and making an angle of 30° with the line through $(5,-2)$ and (2,2).

20. A line through point (5,6) meets a second line at right angles in point $(3,-2)$. Find the equation of the second line.

22. Some general forms of equations of straight lines

The equation of a straight line having y*-intercept* b *and slope* m *is*

$$y = mx + b. \tag{1}$$

This is called the **slope-intercept form.** Its equation (1) is easily deduced by using the method of §**21** and Figure 5. Conversely, any equation having the form (1) must represent a straight line, since it is the equation of the straight line having y-intercept b and slope m.

The equation of a line through a given point (x_1,y_1) *and having slope* m (see Figure 6) *is*

$$y - y_1 = m(x - x_1). \tag{2}$$

This equation is called the **point-slope form.**

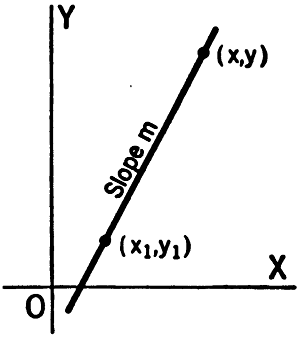


FIG. 6.

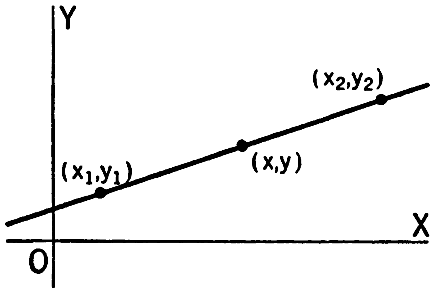


FIG. 7.

The equation of a line through two points (x_1,y_1) *and* (x_2,y_2) (see Figure 7) *is*

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{3}$$

This equation is called the **two-point form.**

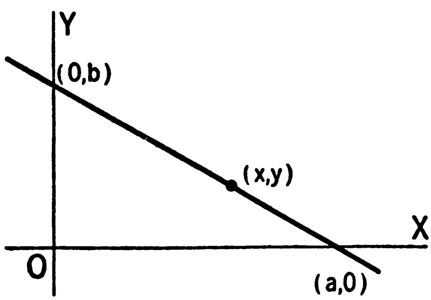


FIG. 8.

The equation of a line having x-*intercept* a *and* y-*intercept* b (see Figure 8) *is*

$$\frac{x}{a} + \frac{y}{b} = 1. \tag{4}$$

This equation is called the **intercept form.**

Equations (2), (3), and (4) are easily derived by using the method of §21 and Figures 6, 7, and 8.

The equation of a line parallel to the y-*axis is*

$$x = a, \tag{5}$$

where (a,b) is a point on it.

Exercises

1. Carry out in detail the derivations of Equations (1), (2), (3), and (4) above.

Using Equation (1), (2), (3), or (4) find the equation of the line specified in each of the exercises numbered 2 to 9:

2. y-intercept 6, slope 3.

3. Through $(2,-3)$, slope 4.

4. Through (2,4) and $(5,-4)$.

5. Having x-intercept 3 and y-intercept -5.

6. Having slope 0, y-intercept 3.

7. Parallel to the y-axis, through (5,6).

8. Passing through (0,5), x-intercept 0.

9. Passing through (5,0), y-intercept 0.

10. Write each of the following equations in the form (1) and give its slope and y-intercept:

(a) $3x + 2y = 12$. (c) $2x - 3y = 7$.
(b) $x - 3y = 9$. (d) $x + 2y = 0$.

11. Write each of the following equations in the intercept form (4) and give its intercepts:

(a) $x + 3y = 6$. (b) $2x - 3y = 7$.

12. Find the equation of a straight line tangent at (4,3) to a circle of radius 5 with center at the origin.

13. Find the equation of the tangent at $(-2,-4)$ to a circle of radius 5 with center at $(2,-1)$.

★**14.** Use the point-slope form (2) to find the equations of the altitudes of the triangle with vertices $(1,-2)$, (3,4), and $(-2,6)$. Show that these three lines meet in a point.*

15. Use the two-point form (3) to find the equations of the medians of the triangle having as vertices (12,6), $(-2,9)$, and $(-4,-6)$. Show that they meet in a point.*

16. Find the intercepts of the line connecting (1,3) and $(-2,4)$.

17. Find the equation of a line whose intercepts are twice those of $2x - 3y = 12$.

18. The point $(3,-2)$ and the midpoint of the line segment cut from $3x - 4y = 12$ by the coordinate axes lie on a certain line. Find its equation.

---

* To show that three lines meet in a point, find the point of intersection of two of them and show that it lies on the third one.

★19. The segment cut from a certain line by the coordinate axes has its midpoint at (5,7). Find the equation of the line.

23. The general equation of the first degree

Theorem. *The general equation of the first degree in* x *and* y:

$$\mathrm{Ax} + \mathrm{By} = \mathrm{C} \tag{6}$$

represents a straight line.

Proof. Equation (6) may be written:

$$y = -\frac{A}{B}x + \frac{C}{B}, \tag{7}$$

provided $B \neq 0$. Comparison of (7) with (1) shows that (7) represents a straight line with slope $-A/B$ and y-intercept C/B. Since every point satisfying either Equation (6) or (7) must satisfy the other, (6) must represent a straight line if $B \neq 0$. If $B = 0$ and $A \neq 0$, Equation (6) may be written

$$x = C/A,$$

and this represents a straight line parallel to the y-axis. A and B are not both zero, since, by hypothesis, Equation (6) is of the first degree.

Comparison of the coefficients of x in (7) and (1) shows that *the slope of*

$$\mathrm{Ax} + \mathrm{By} = \mathrm{C}$$

is $-\mathrm{A/B}$, *or the negative of the coefficient of* x *divided by the coefficient of* y. For example, the slope of $2x + 3y = 10$ is $-\frac{2}{3}$, and the slope of $2y = 3$ is $-\frac{0}{2} = 0$; $2x = 5$, however, has no slope.

The slope relation just discussed is the basis of a short method of finding the equation of a straight line. For example, if a line has slope $\frac{5}{6}$ and passes through point (2,4), we write 5 in front of x and -6 in front of y and equate the result to C, to get

$$5x - 6y = C. \tag{8}$$

This has the proper slope for any value of C. Now we demand that the line pass through point (2,4). Therefore, (2,4) must satisfy (8). Hence, replacing x by 2 and y by 4 in (8), we obtain

$$5(2) - 6(4) = C, \text{ or } C = -14.$$

Substituting -14 for C in (8), we get the required equation:

$$5x - 6y = -14. \tag{9}$$

As another example, let us find the equation of a line perpendicular to (9) and passing through $(1,-3)$. Since the slope of (9) is $\frac{5}{6}$, the required slope is the negative reciprocal of $\frac{5}{6}$, or $-\frac{6}{5}$. Hence, using the method just considered, we have

$$6x + 5y = 6(1) + 5(-3) = -9.$$

Example. Show that an equation of a straight line through (x_1,y_1) and (x_2,y_2) is given in determinant form by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \tag{10}$$

Solution. Equation (10) is linear, as may be seen by expanding it in terms of the elements of the first row and their cofactors. If x is replaced by x_1 and y by y_1 in the first row, two rows become identical; the determinant is therefore zero and the equation is satisfied. Similarly, it is satisfied by the point (x_2,y_2). It follows that (10) represents the line through (x_1,y_1) and (x_2,y_2).

Exercises

1. Write by inspection the slope of:

(a) $2x - y = 5$.
(b) $6x + 10y = 8$.
(c) $x = 10 + 2y$.
(d) $2y = 8 - 3x$.
(e) $4y = 7$.
(f) $2x = 11$.

2. Which of the graphs of the following lines are parallel and which perpendicular?

(I) $3x - 4y = 5$.
(II) $8y - 6x = 11$.
(III) $8x + 6y = 13$.
(IV) $9x = 10 - 12y$.

3. For each of the following lines, find the equation of a parallel line through $(5,-6)$:

(a) $2x + y = 12$. (b) $3x = 7 - 5y$. (c) $2y = 5$.

4. For each of the following lines, find the equation of a perpendicular line through $(-6,3)$:

(a) $x + 3y = 8$. (b) $7x = 5 + 12y$. (c) $2y = 5$. (d) $x = 7$.

5. Show that points $(2,-3)$, $(5,4)$, and $(-1,-10)$ lie in a line by finding the equation of a line through two of them and checking that the third one lies on this line.

6. By finding the point of intersection of two of the lines $x + 3y = 7$, $5x - 6y = -28$, and $4x + 11y = 25$ and showing that it lies on the third line, show that the three lines meet in a point.

7. Use Equation (10) to find the equation of a line through $(1,-2)$ and $(3,0)$.

8. Find the equation of a line from the point on $x + y = 4$ with abscissa 6 to the point on $3x - y = 10$ with ordinate -6.

9. What point on $2x - y = 9$ is equidistant from the origin and $(4,-6)$?

10. Find the coordinates of a point on the y-axis equidistant from $(3,8)$ and $(-2,5)$.

11. Show that the two lines $A_1x + B_1y = C_1$ and $A_2x + B_2y = C_2$ are parallel if $A_1/A_2 = B_1/B_2$ and perpendicular if $A_1A_2 + B_1B_2 = 0$.

12. For what values of a will the lines $ax - 9y = 6$ and $x - ay = 5$ be parallel?

13. For what value of a will the line $x = 6$ be perpendicular to the line $2ax - 8y = 9$?

14. Find the angle that the line $2x + 3y = 6$ makes with the line $x - 2y = 5$.

15. Find the angle that $x + y = 10$ makes with $3x - 2y = 5$.

16. Find the equation of a line through $(2,-2)$ if the sum of its intercepts is 3.

17. Find the equations of two lines parallel to $x - 2y = 25$ each forming with the axes a triangle of area 9.

★18. Find the slopes of all lines through $(1,1)$ which with the axes form in the first quadrant a triangle of area $\frac{9}{4}$.

19. A perpendicular from the origin to a line meets it in $(4,-5)$. Find the equation of the line.

20. A perpendicular from $(2,4)$ to a line meets it in $(3,-1)$. Find the equation of the line.

21. Find the equation of a line through point $(6,6)$ making an angle of $45°$ with $2x + y = 12$.

22. Find the equation of a line through the origin making an angle of $120°$ with $x + y = 10$.

★23. Find the center and radius of a circle circumscribed about the triangle with vertices $(7,4)$, $(3,12)$, and $(5,-2)$.

★24. Show that the feet of the perpendiculars from $(2,1)$ to the lines containing the sides of the triangle with vertices $(4,0)$, $(3,0)$, and $(2,2)$ lie in a straight line.

25. Show that $x + y - 3 + k(2x - y - 15) = 0$ is a line through the intersection of $x + y - 3 = 0$ and $2x - y - 15 = 0$. Determine k if: (a) its y-intercept is -24; (b) it passes through $(5,-8)$; (c) its slope is $\frac{1}{2}$.

26. State the relation between the lines $Ax + By = C$ and $A'x + B'y = C'$ if:

(a) $A/A' = B/B' = C/C'$. (c) $A/A' = B/B' \neq C/C'$.
(b) $AA' + BB' = 0$.

★27. Two points of the curve $y = x^3 - 9x$ have abscissas 1 and 3. Find the equation of the line joining them. Find the coordinates of the third point in which this line intersects the given curve.

24. The normal form of the equation of a straight line

Let OE be a line directed from the origin O to point E (in Figure 9) and making an angle ω with the x-axis. Let l be a

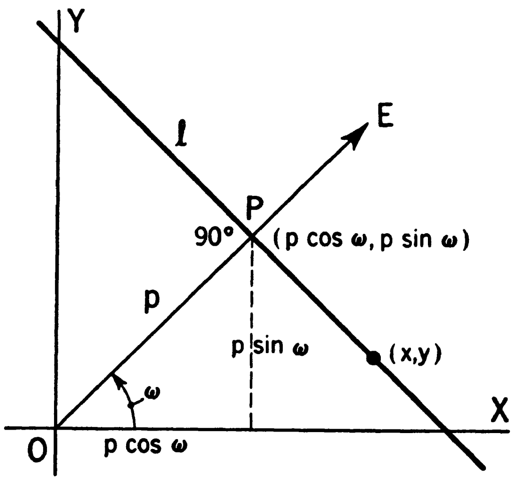


FIG. 9.

line perpendicular to OE having directed distance p from O and meeting OE in P. Then the coordinates of P are $(p \cos \omega, p \sin \omega)$. If (x,y) represents any point on l, the slope of l is $(y - p \sin \omega)/(x - p \cos \omega)$. Also, the line l, being perpendicular to a line of inclination ω, must have as slope the negative reciprocal of $\tan \omega$, or $-1/\tan \omega = -\cos \omega/\sin \omega$. Equating these two values of the slope, we get

$$\frac{y - p \sin \omega}{x - p \cos \omega} = -\frac{\cos \omega}{\sin \omega}, \tag{11}$$

or, clearing of fractions and simplifying,

$$y \sin \omega + x \cos \omega = p(\sin^2 \omega + \cos^2 \omega),$$

$$\boldsymbol{x \cos \omega + y \sin \omega = p.} \tag{12}$$

Distance p may be positive or negative. If ω is so chosen that p is positive, Equation (12) is called the **normal form of the equation of a straight line.** The vector $\overrightarrow{OP}$ (see Figure 9) is called the **normal axis** of the line, and p is called the **normal intercept.**

Figure 10(a) shows the line l corresponding to $\omega = 30°$ and normal intercept $p = 3$. Its equation is

$$x \cos 30° + y \sin 30° = 3. \tag{13}$$

Figure 10(b) shows the line l corresponding to $\omega = 210°$ $p = 3$. Its equation is

$$x \cos 210° + y \sin 210° = 3.$$

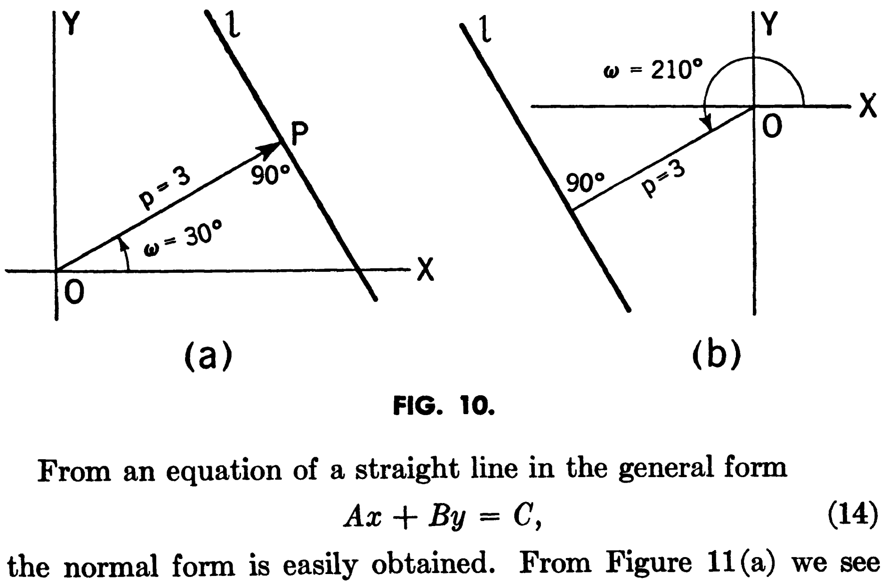


FIG. 10.

From an equation of a straight line in the general form

$$Ax + By = C, \tag{14}$$

the normal form is easily obtained. From Figure 11(a) we see that if A and B are two numbers not both zero, then there is an angle ω such that

$$\cos \omega = \frac{A}{\sqrt{A^2 + B^2}}, \quad \sin \omega = \frac{B}{\sqrt{A^2 + B^2}}; \tag{15}$$

and from Figure 11(b) there is an angle ω' such that

$$\cos \omega' = \frac{A}{-\sqrt{A^2 + B^2}}, \quad \sin \omega' = \frac{B}{-\sqrt{A^2 + B^2}}. \tag{16}$$

Hence, if we divide (14) by that one of the numbers $\sqrt{A^2 + B^2}$ or $-\sqrt{A^2 + B^2}$ which has the same sign as C, take account of (15) and (16), and identify $C/(\pm\sqrt{A^2 + B^2})$ with p, we obtain the

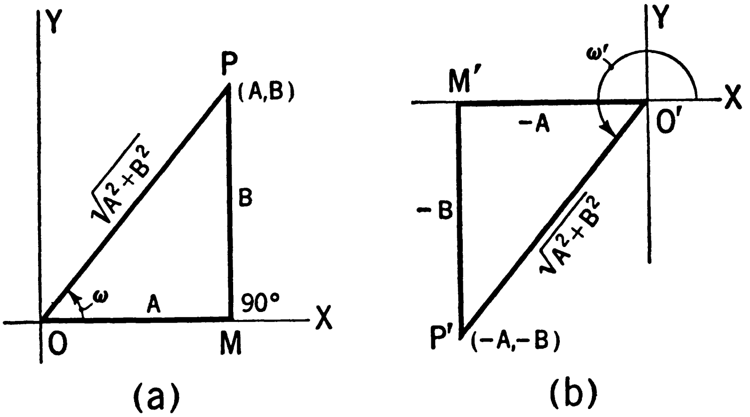


FIG. 11.

equation of the line (14) in normal form. If $C = 0$, take the sign of the divisor the same as that of B so that $\sin \omega = | B/\sqrt{A^2 + B^2} |$ will be positive and ω will be in the range from 0° to 180°. If $C = 0$ and $B = 0$, the normal form is $x = 0$. The following rule summarizes essential facts.

RULE. *To transform an equation*

$$Ax + By = C$$

to normal form, divide through by $\sqrt{A^2 + B^2}$ *or* $-\sqrt{A^2 + B^2}$, *according as* C *is positive or negative; or, if* C = 0, *according as* B *is positive or negative. If* B = C = 0, *the normal form is* x = 0.

For example, to change $3x + 4y = -10$ to normal form, divide through by $-\sqrt{3^2 + 4^2} = -5$ to obtain:

$$-\tfrac{3}{5}x - \tfrac{4}{5}y = 2.$$

Here $p = 2$, $\cos \omega = -\frac{3}{5}$, $\sin \omega = -\frac{4}{5}$, and the graph is shown in Figure 12.

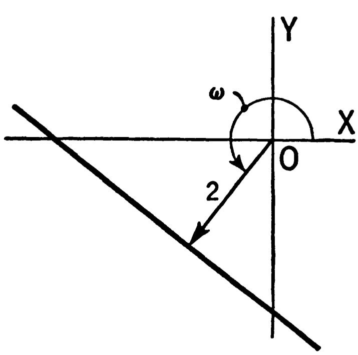


FIG. 12.

Example. The distance of a line from the origin is 5 and it passes through $(-7, -1)$. Find its equation.

Solution. Using (12) with $p = 5$, obtain

$$x \cos \omega + y \sin \omega = 5. \tag{a}$$

Since this must be satisfied by $(-7, -1)$,

$$-7 \cos \omega - \sin \omega = 5, \text{ or } -7 \cos \omega = 5 + \sin \omega. \tag{b}$$

Squaring both members of (b), replacing $\cos^2 \omega$ by $1 - \sin^2 \omega$ in the result, and simplifying, obtain

$$25(\sin \omega)^2 + 5 \sin \omega - 12 = 0. \tag{c}$$

This can be solved by formula 2, page 255, or by factoring. We have

$$25 \sin^2 \omega + 5 \sin \omega - 12 = (5 \sin \omega - 3)(5 \sin \omega + 4). \tag{d}$$

Equating the factors of (d) to zero and solving for $\sin \omega$, we get

$$\sin \omega = \tfrac{3}{5}, \sin \omega = -\tfrac{4}{5}.$$

Substitute $\frac{3}{5}$ for $\sin \omega$ in (b) and solve for $\cos \omega$ to obtain $\cos \omega = -\frac{4}{5}$. Using these values of $\sin \omega$ and $\cos \omega$ in (a), obtain

$$-\tfrac{4}{5}x + \tfrac{3}{5}y = 5.$$

Similarly, using $-\frac{4}{5}$ for $\sin \omega$, obtain $-\frac{3}{5}$ for $\cos \omega$ from (b), and write from (a):

$$-\tfrac{3}{5}x - \tfrac{4}{5}y = 5.$$

Exercises

1. Write each equation in the normal form:

(a) $5x - 12y = 39$.
(b) $2x + 3y = -7$.
(c) $3x + 4y = 0$.
(d) $x - 2y = 0$.
(e) $2y = 3$.
(f) $3x = 4$.
(g) $3x = -4$.
(h) $4x = 0$.

2. Sketch the line $3x - 4y = 12$, showing ω and p on your figure.

3. Find the distance from the origin to each line:

(a) $x + y = 7\sqrt{2}$.
(b) $2x = 3y - 6\sqrt{13}$.
(c) $3x = y$.
(d) $x = 10$.
(e) $y = 3$.
(f) $5 - x = 3y$.

4. Find the distance between the parallel lines:

$$12x - 5y = 13 \text{ and } 12x - 5y = 78.$$

5. Find the distance between the parallel lines:

$$3x - 4y = 32 \text{ and } 3x - 4y = -32.$$

6. Sketch the graphs and write the equations of the lines having the indicated lengths and directions of normal axes:

(a) $p = 10$, $\omega = 45°$.
(b) $p = 10$, $\omega = 135°$.
(c) $p = 10$, $\omega = 225°$.
(d) $p = 10\sqrt{3}$, $\omega = 300°$.
(e) $p = 4$, $\omega = 90°$.
(f) $p = 6$, $\omega = 180°$.

7. Find p and ω for the normal axis of the following lines:

(a) $x + \sqrt{3}y = 10$. (b) $y - x = 7$. (c) $4x = 5$.

8. A perpendicular from the origin to a line meets it at $(-2,3)$. Find the equation of the line in normal form.

9. Find the distances from the origin to the lines:

(a) Through $(2,4)$ and $(5,8)$.
(b) Through $(5,-6)$ and having inclination $60°$.
(c) Having x-intercept 7 and slope 3.

10. Find the equation of a line through $(10,8)$ parallel to $x + 2y = 20$; then find the distances of the two lines from the origin, and the distance between them.

11. Find two values of K such that for each the line $Kx + y = 5$ is distant 4 from the origin.

★**12.** Find the equation of a line having $p = 2\sqrt{2}$ and passing through $(1,3)$.

★**13.** Find the equations of the lines:

(a) Distant 5 from the origin and passing through $(7,1)$.
(b) Distant $\sqrt{13}$ from the origin and passing through $(5,-1)$.

25. Distance from a line to a point

The distance d from a line l to a point (x_1,y_1) (see Figure 13) is easily found. Refer line l, represented by (12), to parallel axes through (x_1,y_1) by the translation

$$x = x' + x_1,\ y = y' + y_1, \tag{17}$$

and transform slightly to obtain:

$$(x' + x_1)\cos\omega + (y' + y_1)\sin\omega = p, \tag{18}$$

$$x'\cos\omega + y'\sin\omega = p - x_1\cos\omega - y_1\sin\omega. \tag{19}$$

Here the right member is the distance from the new origin (x_1,y_1) to line (18) and, therefore, to line (12). The distance from line (12) to (x_1,y_1) is the same thing with sign reversed; therefore,

$$\boldsymbol{d = x_1\cos\omega + y_1\sin\omega - p.} \tag{20}$$

This formula is the justification for the following rule.

RULE. *To find the distance* d *from a line*

$$\mathrm{Ax} + \mathrm{By} = \mathrm{C} \tag{21}$$

to a point $(\mathrm{x}_1,\mathrm{y}_1)$, *write the equation of the line in normal form with all terms in the left member, and replace* x *by* x_1 *and* y *by* y_1 *in the result to obtain*

$$\mathbf{d} = \frac{\mathbf{Ax}_1 + \mathbf{By}_1 - \mathbf{C}}{\pm\sqrt{\mathbf{A}^2 + \mathbf{B}^2}}, \tag{22}$$

where the sign of the denominator is to be taken the same as the sign of C.

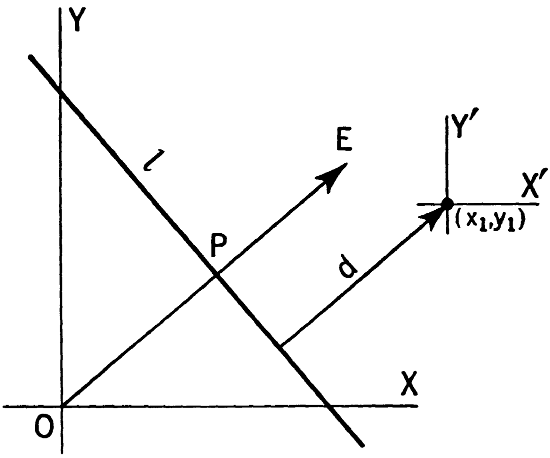


FIG. 13.

Note that, except for the case where the line passes through the origin, the positive direction for distance d is the direction of the normal axis. Hence, d given by (20) or (22) is positive when point (x_1,y_1) is on the side of the line not containing the origin, and negative when (x_1,y_1) is on the same side of the line as the origin.

Consider, for example, the line

$$3x + 4y = 20$$

and the points $(-1,3)$ and $(6,5)$. Using (22), we have for

$$(-1,3),\ d = \frac{-3 + 12 - 20}{5} = -\frac{11}{5},$$

$$(6,5),\ d = \frac{18 + 20 - 20}{5} = \frac{18}{5}.$$

Figure 14 shows the given line, the given points, and their distances from the line.

Example. Find the equations of the bisectors of the angles having $x + 2y = 10$ and $2x + y - 8 = 0$ as sides.

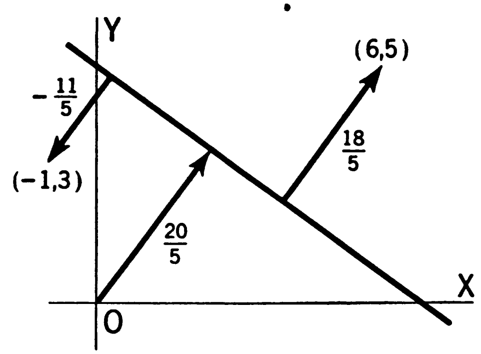


FIG. 14.

Solution. Since every point (x_1,y_1) on the bisector of an angle is equidistant from the sides, we must have

$$\frac{x_1 + 2y_1 - 10}{\sqrt{5}} = \pm \frac{2x_1 + y_1 - 8}{\sqrt{5}},$$

or, using first the plus sign and then the minus sign in the right member and simplifying,

$$-x_1 + y_1 - 2 = 0, \; 3x_1 + 3y_1 - 18 = 0, \tag{a}$$

or, dropping the subscripts and simplifying,

$$x - y + 2 = 0, \; x + y - 6 = 0. \tag{b}$$

Figure 15 shows the two given lines AD and BC and the lines bisecting the angles between them. Observe that the first equa-

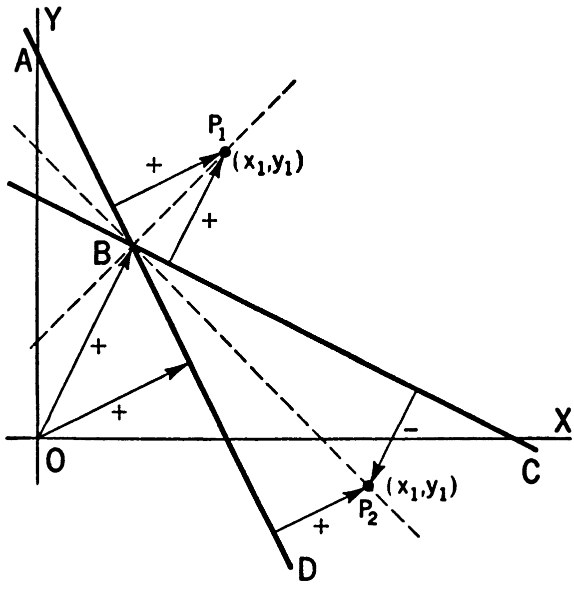


FIG. 15.

tion of (a), obtained by considering equal distances, represents the bisector of angle ABC; and the second equation, obtained by considering distances equal numerically but opposite in sign, represents the bisector of angle DBC.

Exercises

1. Find the directed distances from the given lines to the indicated points by using (22):

(a) $3x + 4y - 10 = 0$; $(6,-7)$.
(b) $4x - 3y + 25 = 0$; $(-2,8)$.
(c) $x + y = 12$; $(-2,4)$.
(d) $x + y = -4$; $(5,3)$.
(e) $x + 3y = 0$; $(1,-7)$.
(f) $y = 3$; $(3,-3)$.

2. For the triangle with vertices $A(5,-3)$, $B(4,4)$, and $C(-3,1)$, find the length of the altitude from A to side BC and then find the area of the triangle.

3. Find the area and the length of the altitude to side AB of the triangle with vertices $A(1,-3)$, $B(-3,4)$, and $C(-6,-4)$.

4. Show that $3x + y - 10 = 0$ and $x + 3y = 6$ are tangents to a certain circle having its center at $(5,3)$.

5. Show that $y + 4x = 25$ and $4x - y + 1 = 0$ are tangents to a circle having $(3,-5)$ as center.

6. Is $(2.98, 4.02)$ on the same side of $3x + 4y = 25$ as the origin?

7. Find the distance between the parallel lines:

$$5x + 6y = 25 \text{ and } 5x + 6y = 72.$$

8. Find the distance between the parallel lines:

$$3x - 2y = 27 \text{ and } 3x - 2y = -43.$$

9. Find the equations of two tangents to a circle with center at the origin and radius 10, each having as slope $-\frac{3}{4}$.

10. Does the line $2x - y = 5$ cut the circle with center $(2,3)$ and radius 2.1 in two real points?

11. If the directed distance from $3x + 4y = 25$ to $(2,h)$ is 9, find h.

12. Find two points each with abscissa 2 and distant 4 units from $5x - 12y = 65$.

13. Find the equation of the locus of points equidistant from $x + 2y = 10$ and $x + 2y = 25$.

14. Find the locus of points having directed distance -15 from $3x + 4y = 20$ and also the locus of points distant 5 from $3x + 4y = 20$.

15. Find k if the directed distance from $3x + ky = 10$ to $(3,-2)$ is -2.

16. Find k if the distance (undirected) from $kx + 4y = 2$ to $(2,4)$ is 4.

In Exercises 17 and 18, find the bisectors of the angles formed by the given lines.

17. $8x + 15y = 51$, $15x - 8y = 85$.

18. $3x + 4y = 5$, $5x - 12y = 26$.

★19. Find the coordinates of the point where the bisector of angle A meets side BC of the triangle having vertices $A(1,2)$, $B(4,-2)$, and $C(-5,-6)$.

★20. Find the coordinates of the point where the bisector of angle A meets side BC of the triangle having vertices $A(-1,4)$, $B(10,2)$, and $C(2,-2)$.

★21. Find the center and radius of the circle inscribed in the triangle with sides along $x = 0$, $y = 0$, and $3x + 4y - 48 = 0$.

★22. Find the center and radius of a circle inscribed in a triangle having sides along the lines $3x + 4y - 9 = 0$, $4x + 3y - 5 = 0$, and $3x - 4y = 1$.

23. Two vertices of a triangle are $(2,3)$ and $(-6,9)$. If the area of the triangle is 30, find the locus of the third vertex.

24. Find the locus of a point equidistant from point $(0,0)$ and line $x + y = 4$.

25. Find the locus of a point twice as far from $(0,0)$ as from $x + y = 4$.

26. Find the locus of a point always twice as far from $x + y = 10$ as from the y-axis.

27. Find the locus of a point always three times as far from $3x + 4y = 10$ as from $4x - 3y = 30$.

★28. Prove that the sum of the lengths of the perpendiculars from a point on the base of an isosceles triangle to its equal sides is constant.

★29. Prove that the bisector of an angle of a triangle divides the opposite side into two parts which have the same ratio as the other two sides.

30. Define the positive direction for distance given by (22) when: (a) $C = 0$, $B \neq 0$; (b) $B = C = 0$.

26. Systems of lines

The equation

$$y = 2x + b \tag{23}$$

represents all lines having slope 2. Figure 16 shows lines corresponding to a few values of b. To every value of b corresponds a straight line.

A set of lines defined by a linear equation in x and y having one and only one arbitrary constant is called a *system of lines*, often a *family of lines*. The arbitrary constant is called a *parameter*. Generally, each line of a system has some specific property. Thus, every line in the system represented by

$$x \cos \alpha + y \sin \alpha = 5$$

is distant 5 from the origin.

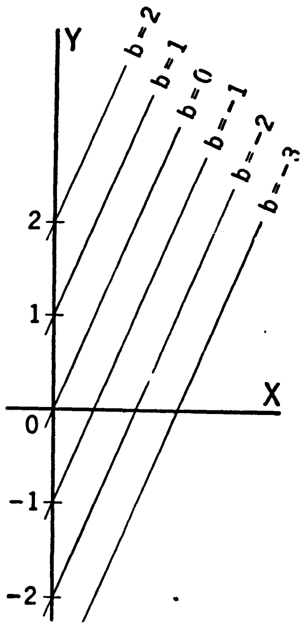


FIG. 16.

Example. Write the equations of the systems of lines characterized by the following properties:

(a) Parallel to $3x + y = 10$.
(b) Perpendicular to $3x + y = 10$.
(c) Passing through (2,3).
(d) Distant 5 from (0,0).

Solution. The student can easily verify the following answers:

(a) $3x + y = k$.
(b) $x - 3y = k$.
(c) $y - 3 = m(x - 2)$.
(d) $x \cos \omega + y \sin \omega = 5$.

Exercises

Write the equations of the systems of lines possessing the properties numbered 1 to 10:

1. Passing through the origin.
2. Passing through $(-3,2)$.
3. Distant 10 from the origin.
4. Parallel to $x + y = 10$.
5. Perpendicular to $3x + 2y = 7$.
6. Having x-intercept 6.
7. Having inclination 30°.
8. Having the algebraic sum of its intercepts 10.
9. Bounding with the axes a triangle of area 10 in the first quadrant.
10. Having as inclination twice that of $x = 2y$.

For each of the systems represented by the equations numbered 11 to 16, state a geometric property which characterizes it:

11. $5x + 6y = k$.

12. $y = kx + k$.

13. $x + ky = 5$.

14. $(x/k) + y/(1 - k) = 1$.

15. $kx + \sqrt{1 - k^2}\, y = 20$.

16. $(x - 2) \cos \omega + (y + 3) \sin \omega = 10$.

17. In each of the systems numbered 11 to 14, determine k for the member through $(2, -1)$.

★**18.** Each line of a system has the sum of its x-intercept and y-intercept equal to 4. Write the equation of the system and find the x-intercept of those members forming with the axes a triangle of area $\frac{5}{2}$.

★**19.** Write the equation of the system of lines distant 3 from the origin. Find the y-intercepts of those members of the system passing through (1,3).

20. Show that $x + y + k(2x - y + 4) = 0$ represents a system of lines through the intersection of $x + y = 0$ and $2x - y + 4 = 0$. Determine the value of k for the member having 8 as y-intercept.

21. Solve Exercise 20 and then prove that if $l_1 = 0$* and $l_2 = 0$ are the equations of two straight lines, then $l_1 + kl_2 = 0$ is the equation of a system of lines through the intersection of the lines $l_1 = 0$ and $l_2 = 0$.

In accordance with the principle of Exercise 21, $x + 2y - 5 + k(2x + 3y - 4) = 0$ is the equation of a system of lines through the intersection of $x + 2y - 5 = 0$ and $2x + 3y - 4 = 0$. For each of the conditions numbered 22 to 29, find k for the member of the system satisfying the condition:

22. x-intercept 3.

23. y-intercept 1.

24. Slope $-\frac{3}{4}$.

25. y-intercept equal to x-intercept.

26. Passing through (1,1).

27. Parallel to $3x - 4y = 10$.

28. Perpendicular to $2x + y = 10$.

29. Distant $\sqrt{2}$ from the origin.

30. Find the equations of the system of lines having equal intercepts.

* The notation $l_1 = 0$ is shorthand for $A_1x + B_1y + C_1 = 0$.

CHAPTER V

The Circle

27. The equation of a circle

To find the equation of a circle with center (h,k) and radius a (see Figure 1), let (x,y) be any point on the circle, and, using Formula (3), §15, express the condition that distance from center (h,k) to (x,y) is a. This gives as *the equation of a circle with center* (h,k) *and radius* a

$$\sqrt{(x-h)^2+(y-k)^2}=a$$

or

$$(x-h)^2+(y-k)^2=a^2. \quad (1)$$

For example, the equation of a circle with center (2,0) and radius 3 is

$$(x-2)^2+y^2=9.$$

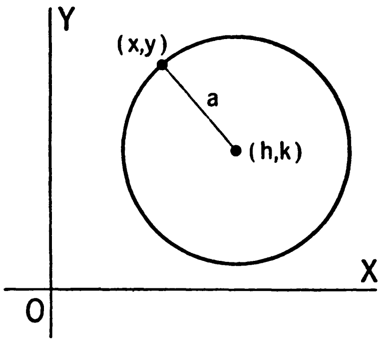


FIG. 1.

In general, *to get the equation of a circle, find its radius and the coordinates of its center, and then use Equation* (1).

Example. Find the equation of a circle through (0,0) and having 4 as x-intercept and -2 as y-intercept.

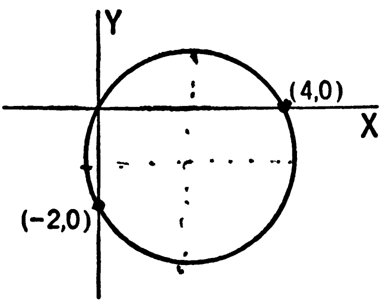


FIG. 2.

Solution. Figure 2 represents the required circle. Inspection of the figure shows that $h=\frac{1}{2}(4)=2$, and $k=\frac{1}{2}(-2)=-1$. Substituting these values for h and k in (1), we obtain

$$(x-2)^2+(y+1)^2=a^2. \quad \text{(a)}$$

Since the circle must pass through the origin, (0,0) must satisfy (a). Hence,

$$(0-2)^2+(0+1)^2=a^2, \text{ or } a^2=5.$$

Replacing a^2 by 5 in (a), we have

$$(x - 2)^2 + (y + 1)^2 = 5.$$

Exercises

1. Find the equations of the circles having the indicated centers and radii:

(a) Center (0,0), radius 3.
(b) Center (0,5), radius 4.
(c) Center (−5,0), radius 5.
(d) Center (2,−3), radius 4.

Find the equations of the circles having the properties indicated in Exercises 2 to 9.

2. Center (3,1), and passing through (0,5).

3. x-intercept 4, y-intercept 2, and passing through (0,0).

4. Passing through (0,0), (0,8), and (6,0).

5. Center on $x = 3$, radius $\sqrt{13}$, and passing through (6,5).

6. Center on the x-axis, radius 5, and passing through (0,0). Find the equations of two circles.

7. Center on the y-axis, radius 5, and having 3 as an x-intercept.

8. Tangent to the x-axis at (5,0) and having radius 3.

9. Tangent to the y-axis at (0,3) and passing through (2,5).

10. Find two points on the y-axis each distant 5 from the point (3,6).

11. Find two points on the line $x + y = 2$ each 5 units from (0,3).

12. Find the equation of a circle having as end points of a diameter: (a) (3,−2), (5,−4); (b) (4,0), (0,6); (c) (0,0), (−2,−6).

13. The equation $(x - h)^2 + y^2 = 25$ represents the system of circles with center on the x-axis and radius 5. Describe the system of circles represented by each of the following equations:

(a) $x^2 + (y - k)^2 = 16$.
(b) $x^2 + y^2 = a^2$.
(c) $(x - a)^2 + (y - a)^2 = 36$.
(d) $(x - 2)^2 + (y + 3)^2 = a^2$.
(e) $(x - h)^2 + (y - k)^2 = 25$.
(f) $(x - a)^2 + (y - a)^2 = a^2$.

Transform each of the equations numbered 14 to 17 by translation of axes to the indicated origin, and draw both sets of axes and the graph of the equation:

14. $x^2 + 2x + y^2 - 4y = 4$; $(-1,2)$.

15. $x^2 + y^2 - 4x = 0$; $(2,0)$.

16. $2x^2 + 2y^2 + 3x - 5y = 15$; $(-\frac{3}{4},\frac{5}{4})$.

17. $x^2 + 2ax + y^2 + 2by = c$; $(-a,-b)$. Omit the graph.

28. The general equation of a circle

If Equation (1), §27, were expanded, the coefficients of x^2 and y^2 in the result would both be 1. *Conversely,* as we shall show later, *any equation having the form*

$$\mathrm{x}^2 + \mathrm{y}^2 + \mathrm{Ax} + \mathrm{By} + \mathrm{C} = 0 \tag{2}$$

will represent a circle if $\mathrm{A}^2 + \mathrm{B}^2 > 4\mathrm{C}$.

Consider, for example,

$$2(x^2 + y^2) + 4x + 3y = 10. \tag{3}$$

Let us attempt to change it to the form (1). Dividing through by 2 and rearranging slightly, we get

$$(x^2 + 2x \qquad) + (y^2 + \tfrac{3}{2}y \qquad) = 5. \tag{4}$$

Now, $x^2 + 2x + [\frac{1}{2}(2)]^2$ is a perfect square, and $y^2 + \frac{3}{2}y + [(\frac{1}{2})(\frac{3}{2})]^2$ is a perfect square. Hence, add $1^2 + (\frac{3}{4})^2$ to both members of (4) and obtain

$$(x^2 + 2x + 1) + (y^2 + \tfrac{3}{2}y + \tfrac{9}{16}) = 5 + 1 + \tfrac{9}{16} = \tfrac{105}{16}, \tag{5}$$

or

$$(x + 1)^2 + (y + \tfrac{3}{4})^2 = \tfrac{105}{16}. \tag{6}$$

Evidently this represents a circle with radius $\sqrt{105}/4$ and center $(-1,-\frac{3}{4})$.

Now apply the same process to (2). Rearranging slightly and adding $(A/2)^2 + (B/2)^2$ to both members, obtain

$$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 = -C + \frac{A^2}{4} + \frac{B^2}{4}. \tag{7}$$

Comparing this with (1), §27, we perceive that it represents a circle with center $(-A/2,-B/2)$ and radius $\frac{1}{2}\sqrt{A^2 + B^2 - 4C}$.

In order that a circle be real, the square of its radius must be positive. Hence, it is apparent from (7) that if

$$-C + \frac{A^2 + B^2}{4} \leqq 0, \text{ or } A^2 + B^2 \leqq 4C,$$

the radius will be imaginary or zero and there will be no circle.*

* The locus of $(x - h)^2 + (y - k)^2 = 0$ consists only of the point (h,k); this is often called a point circle, but, strictly speaking, it is not a circle.

The student should make no effort to memorize any of these formulas. *To find the center and radius of a circle, change its equation to the form* (1), §**27**, *and read center and radius from the transformed equation.*

Example 1. Find the equation of a circle with radius twice that of circle $x^2 + y^2 + 4x - 6y = 23$ and concentric with this circle.

Solution. Transforming the given equation to the form (1), §**27**, obtain

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) = 23 + 4 + 9 = 36,$$

or

$$(x + 2)^2 + (y - 3)^2 = 36.$$

Since the radius of this circle is 6 and its center is $(-2,3)$, the required circle has radius 12 and the same center. Hence, its equation is

$$(x + 2)^2 + (y - 3)^2 = 144.$$

Example 2. Show that $x^2 + y^2 - 4x - 8y + 25 = 0$ does not represent a real circle.

Solution. Transforming the given equation to form (1), §**27**, obtain

$$(x - 2)^2 + (y - 4)^2 = -5.$$

Since the square of the radius is negative, no real circle is represented.

Exercises

Find the centers and radii and draw the graphs of the circles defined by the equations numbered 1 to 6.

1. $x^2 + y^2 - 2x - 8y = 8.$

2. $x^2 + y^2 + 3x = 4.$

3. $4x^2 + 4y^2 - 16x + 24y + 27 = 0.$

4. $x^2 + y^2 + 4x = 0.$

5. $x^2 + y^2 - 7y = 0.$

6. $x^2 + y^2 + 2mx + 2ny = 0.$

7. Find the equation of a circle concentric with $x^2 + y^2 + 2x - 4y = 0$ and passing through (5,3).

8. Find the equation of a circle concentric with $x^2 + y^2 - 5x - 7y = 10$ and tangent to $x + y = 10$.

9. Determine which of the following equations represents a circle with real radius different from zero:

(a) $x^2 + y^2 + 10x + 30 = 0$.
(b) $3x^2 + 3y^2 - 11x + 91 = 0$.
(c) $4x^2 + 4y^2 + 18x - 8y + 85 = 0$.
(d) $36x^2 + 36y^2 - 36x + 48y + 16 = 0$.

10. State two conditions which must hold if the equation

$$ax^2 + 2y^2 + cx + dy + e = 0$$

is to represent a circle with real radius different from zero.

11. Find the center and radius of the locus of points situated twice as far from (6,0) as from (0,0).

12. Find the center and radius of the locus of points for which the square of the distance from (4,0) is numerically equal to three times the distance from the y-axis.

13. Find the equation of a circle through the origin and having a and b as x- and y-intercepts, respectively.

14. Find the equation of a circle with center on $x + 2y = 0$ and passing through (4,3) and $(-1,-2)$.

15. Is there a circle on which lie the four points (0,0), (6,0), $(0,-4)$, and (5,1)?

16. Find the locus of midpoints of chords connecting the origin to points on the circle: (a) $x^2 + y^2 = 8x$. (b) $x^2 + y^2 = 8x - 4y$.

17. Find the locus of a point which moves so that the sum of the squares of the perpendiculars from the point to the sides or sides prolonged of a rectangle is constant. Take the axes of coordinates along two adjacent sides of the rectangle.

18. Find the equation of the locus of a point (x,y) at which the line segment connecting (a,a) and $(-a,a)$ subtends a right angle.

19. Find the equation of the locus of a point (x,y) at which the line segment connecting $(\pm a,0)$ subtends an angle of: (a) 30°. (b) 135°.

20. Show that the locus of a point k times as far from (0,5) as from (0,0) is a circle provided $k \neq 1$.

21. Find the center and radius of the locus of a point twice as far from (0,3) as from (0,0).

22. Find the center and radius of the locus of a point having the sum of the squares of its distances from $x + y = 2$ and $x - y = 3$ equal to 3.

23. The end A of a line segment of length $2l$ moves on the x-axis and the other end B moves on the y-axis. Find the locus of the midpoint of the segment.

24. The angle subtended at a moving point P by a line segment with ends at $(\pm m,0)$ is constant. Prove that the locus of P consists of two circular arcs.

29. Equations of circles satisfying three conditions

To solve problems relating to circles, it is a good plan, after drawing a figure representing the required circle and essential data, to set up three equations in the three unknowns h and k for the center (h,k) and a for the radius, solve them simultaneously for h, k, and a, and then use (1), §27. An important part of the solution consists in setting up these equations. If, for example, the circle passes through (2,3), then

$$(2 - h)^2 + (3 - k)^2 = a^2.$$

If the center of the circle lies on $2x + 4y + 5 = 0$, then

$$2h + 4k + 5 = 0.$$

If the circle is tangent to $2x + 3y = 8$ at (1,2), then (h,k) lies on the line perpendicular to $2x + 3y = 8$ at (1,2), namely, $3x - 2y = -1$, and

$$3h - 2k = -1.$$

Also, point (1,2) lies on the circle, and

$$(1 - h)^2 + (2 - k)^2 = a^2.$$

Since the distance from (h,k) to $2x + 3y = 8$ is a, we could also use

$$\frac{2h + 3k - 8}{\sqrt{13}} = \pm a.$$

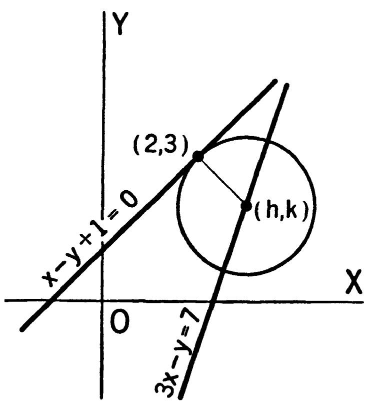


FIG. 3.

The following examples will illustrate the procedure.

Example 1. Find the equation of a circle with center on $3x - y = 7$ and tangent to $x - y + 1 = 0$ at (2,3).

Solution. Figure 3 shows the required circle and essential relations. Since the center (h,k) lies on $3x - y = 7$,

$$3h - k = 7. \qquad \text{(a)}$$

As $x - y + 1 = 0$ is tangent to the circle at (2,3), the perpendicular to $x - y + 1 = 0$ at (2,3), namely, $x + y = 5$, passes through (h,k) and

$$h + k = 5. \qquad \text{(b)}$$

Also, the distance from (h,k) to $(2,3)$ is equal to the radius a, and, therefore,

$$(2 - h)^2 + (3 - k)^2 = a^2. \tag{c}$$

The solution of (a) and (b) for h and k is $h = 3$, $k = 2$. Substituting these values in (c), we get

$$(2 - 3)^2 + (3 - 2)^2 = a^2, \text{ or } a^2 = 2.$$

Therefore, the required equation is

$$(x - 3)^2 + (y - 2)^2 = 2. \tag{d}$$

Example 2. Find the equation of a circle tangent to the y-axis and passing through $(1,2)$ and $(2,3)$.

Solution. Figure 4 shows a circle satisfying the given conditions. Observe that $h = a$. Hence, each of the points $(1,2)$ and $(2,3)$ satisfies (1), §**27**, with a replaced by h. Therefore,

$$(1 - h)^2 + (2 - k)^2 = h^2,$$
$$(2 - h)^2 + (3 - k)^2 = h^2.$$

To solve these equations, subtract one from the other, member by member, to get a linear equation, and then solve this simultaneously with either one. The result is two solutions,

$$h = 1, k = 3 \text{ and } h = 5, k = -1.$$

Hence, remembering that the radius a equals h, write from (1), §**27**, two answers:

$$(x - 1)^2 + (y - 3)^2 = 1,$$
$$(x - 5)^2 + (y + 1)^2 = 25.$$

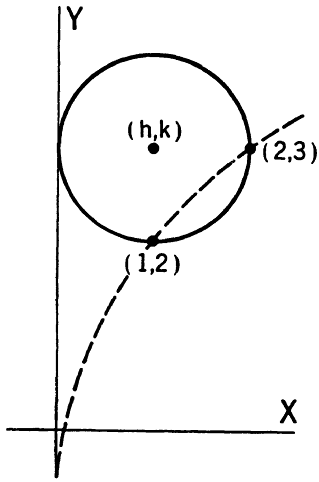


FIG. 4.

The dotted circle in Figure 4 represents the second solution.

Exercises

Write equations connecting h, k, and a corresponding to the conditions numbered 1 to 9:

1. A circle passes through $(1,-2)$.

2. A circle has its center on $x - 2y = 3$.

3. A circle is tangent to the x-axis.

4. A circle is tangent to the y-axis at $(0,4)$.

5. A circle has x-intercepts 0 and 4.

6. A circle is tangent to $3x + 4y - 5 = 0$.

7. A circle is tangent to $3x + 4y - 5 = 0$ at $(1,\frac{1}{2})$.

8. A circle passes through A and B and is tangent to BC at B, where the points referred to are $A(0,0)$, $B(2,-1)$, and $C(4,1)$. (Three equations.)

9. A circle passes through $(1,1)$, $(2,2)$, and $(0,4)$. (Three equations.)

Find the equations of the circles satisfying the conditions stated in the exercises numbered 10 to 21:

10. Tangent to the x-axis at $(5,0)$ and having its center on the line $x + 2y = 11$.

11. Tangent to the y-axis at $(0,3)$ and passing through $(2,1)$.

12. Tangent to both axes and having its center on the line $y = 2x - 6$.

Hint. $h = k$ for one solution and $h = -k$ for another.

13. Center $(2,3)$ and tangent to $x + y = 20$.

14. Abscissa of center 6 and tangent to $3x + 4y = 25$ at $(3,4)$.

15. Tangent to $y = 0$ and to $4x - 3y = 0$ and passing through $(2,2)$. (Two answers.)

16. Tangent to $2x + y = 5$ at $(2,1)$ and having its center on $x + y = 6$.

17. Tangent to $x^2 + y^2 = 2$ at $(1,1)$ and having radius $\sqrt{18}$. (Two answers.)

18. Having abscissa of center 2, tangent to $x^2 + y^2 = 1$, and having 2 as radius.

19. Tangent to $y = x$ at $(2,2)$ and to the x-axis.

20. Tangent to the y-axis and passing through $(4,1)$ and $(4,5)$.

21. Tangent to the x-axis and passing through $(2,1)$ and $(3,2)$.

Find the equations of the circles passing through the sets of three points in the exercises numbered 22 to 25:

22. $(0,0)$, $(4,0)$, $(6,2)$.

23. $(2,0)$, $(0,-4)$, $(-1,-1)$.

24. $(3,1)$, $(1,5)$, $(-1,-1)$.

25. $(-4,-2)$, $(1,3)$, $(-5,-1)$.

26. Find the equation of a circle inscribed in a triangle having its sides along $x = 0$, $y = 0$, and $3x - 4y = 16$.

27. Prove that the following equation represents a circle through points (a,b), (c,d), and (e,f):

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ a^2 + b^2 & a & b & 1 \\ c^2 + d^2 & c & d & 1 \\ e^2 + f^2 & e & f & 1 \end{vmatrix} = 0.$$

★**28.** Let $l_1 = 0$, $l_2 = 0$, $l_3 = 0$ be the equations in normal form of the sides of a triangle. If the origin is inside, as in Figure 5, the equations of the bisectors of the angles are

$$l_1 = l_2,\ l_1 = l_3,\ l_2 = l_3.$$

The point of intersection of the first two bisectors must satisfy the equation of the third, since the third is obtained by subtracting the first two, member by member. Hence, the bisectors meet in a point. Make the argument for a case where the origin is not inside the triangle.

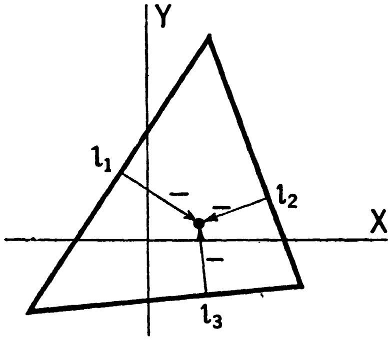


FIG. 5.

★**29.** Show that, for the triangle of Figure 5, the equation

$$Al_1l_2 + Bl_2l_3 + l_1l_3 = 0$$

will be a circle if A and B are so chosen that the coefficient of x^2 equals that of y^2 and so that the coefficient of xy is zero. Show that this circle will pass through the vertices of the triangle and will, therefore, be the equation of the circumscribed circle. Use this principle to find the equation of the circle circumscribed about the triangle with sides along $x = 0$, $y = 0$, and $2x + 3y = 12$.

★**30.** Find the locus of points P for which the feet of the perpendiculars from P to the three lines $x = 0$, $y = 0$, and $3x + 4y = 24$ lie in a straight line. Sketch the three lines and the required locus.

30. Systems of circles

The equation

$$x^2 + y^2 + 6x - 2y - 16 + k(x^2 + y^2 - 8) = 0 \tag{8}$$

represents, for each value of k except -1, a circle through the points of intersection of the circles

$$x^2 + y^2 + 6x - 2y - 16 = 0 \text{ and } x^2 + y^2 - 8 = 0. \tag{9}$$

Equation (8) represents a circle because it can be written in the general form (2), §28; and it passes through the points of intersection of the circles (9), since for such points Equation (8) becomes $0 + k\,0 = 0$. For example, (2,2) satisfies the two equations (9) and therefore satisfies (8). For one particular value of k ($k = -1$), (8) reduces to the equation of a straight line, namely,

$$6x - 2y - 8 = 0. \tag{10}$$

Since this line passes through the points of intersection of the circles (9), it is the common chord of the circles. In Figure 6, the line (10) and several of the circles represented by (8) are shown.

In general, if

$$u = 0, v = 0 \tag{11}$$

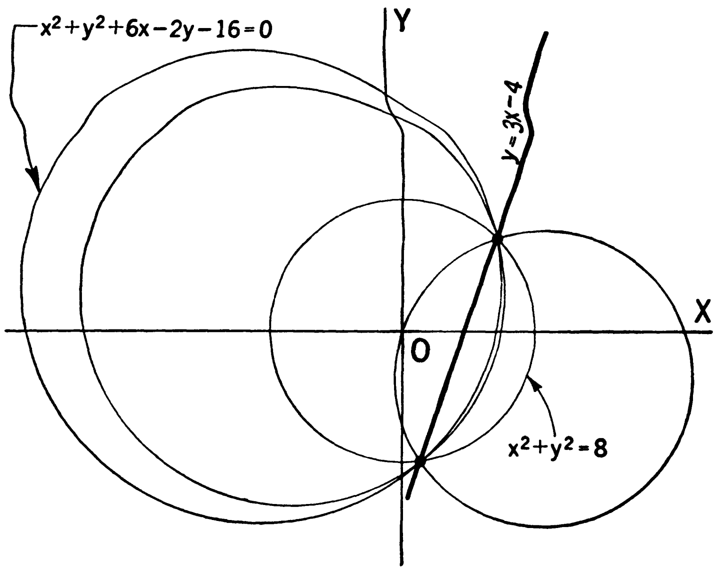


FIG. 6.

are the equations in form (2), §28, of two circles, then

$$u + kv = 0 \tag{12}$$

represents a system of circles, each one satisfied by the simultaneous solutions of (11).

The straight line represented by

$$u - v = 0^* \tag{13}$$

is called the **radical axis** of the circles (11). It is their common chord if they intersect. Various relations of circles will be considered in the following list of problems, and Exercises 13 to 16 relate to several kinds of systems represented by (12).

* No straight line is represented when the given circles are concentric.

Exercises

1. Show that circles (9) intersect in points (2,2) and $(\frac{2}{5}, -\frac{14}{5})$ and that these points satisfy (8) and (10).

2. Determine k in Equation (8) if: (a) the graph has 2 as x-intercept; (b) the graph passes through (3,3).

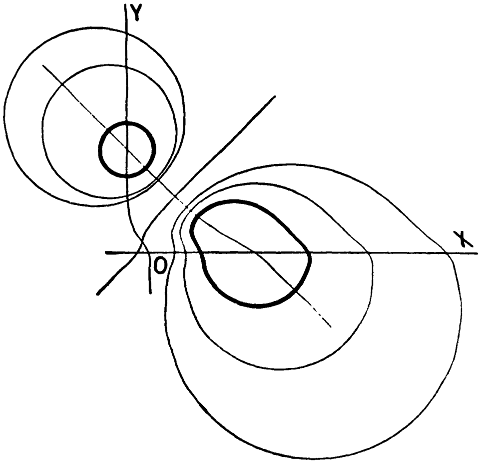


FIG. 7.

3. Find the angle of intersection of the circles (9). The angle between two curves is the angle between their tangents at the point of intersection.

4. Write the equation of the system of circles passing through the intersections of $x^2 + y^2 - 5 = 0$ and $x^2 + y^2 - 4x - 5 = 0$, and the equation of the radical axis of the given circles.

5. Figure 7 shows several circles of the system

$$x^2 + y^2 - 8x + 12 + k(x^2 + y^2 - 8y + 15) = 0$$

and the radical axis

$$-8x + 8y - 3 = 0$$

of the two basic circles. Find the equation of that circle of the system through: (a) (0,0). (b) (0,2).

6. Show that the three radical axes of three circles taken in pairs meet in a point.

Hint. If $u = 0$, $v = 0$, and $w = 0$ are the equations of the circles in form (2), **§28**, then the equations of the radical axes are $u = v$, $u = w$, and $v = w$.

7. Figure 8 shows a circle with center (h,k) and radius a, a point $P(m,n)$ outside it, and the point of contact A of a tangent from P to the circle. Prove that

Length $PA = \sqrt{(m-h)^2 + (n-k)^2 - a^2}$.

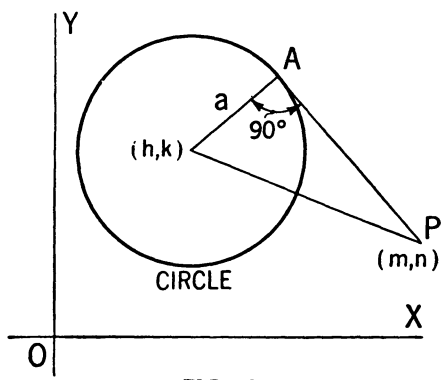


FIG. 8.

Using the formula of Exercise 7, find the length indicated by PA in Figure 8 for each circle and point numbered 8 to 10:

8. $(x-3)^2 + (y-2)^2 - 7 = 0$; $P(7,6)$.

9. $(x - 1)^2 + y^2 - 25 = 0$; $P(6,12)$.

10. $x^2 + y^2 - 8 = 0$; $P(6,6)$.

11. Prove that the distances measured along the tangents from any point on the radical axis of two circles to the points of contact with them are equal.

12. Prove that the radical axis of two circles is perpendicular to the line connecting their centers.

Hint. Take the origin at the center of one circle and the x-axis along their line of centers.

13. Show that if the circles (11) intersect in two real points, then every circle through those two points is represented by (12).* To do this, show that through every point (x_1, y_1) in the plane, not on the circle $v = 0$ nor on the line through the two points of intersection, there passes a circle represented by (12).

14. If the circles (11) are tangent to a line $ax + by = c$ at point (m,n) on the line, what system is represented by (12)?

15. If the circles (11) are concentric, what system is represented by (12)?

★16. Show that every circle of the system (12) cuts $x^2 + y^2 = c^2$ at right angles if

$$u = x^2 + y^2 - 2ax + c^2,\ v = x^2 + y^2 - 2bx + c^2,\ a^2 > b^2 > c^2.$$

Remark. In this case (12) represents any system of circles based on two non-intersecting and non-concentric circles.

* Actually, the circle $v = 0$ is not represented by (12).

CHAPTER VI

Polar Coordinates

31. Foreword

There is no limit to the number of methods of associating equations to graphs. Perhaps the most important method is based on the rectangular coordinates used up to this point. However, the methods considered in this chapter and in the next are of great importance. Some problems easily solved by using polar coordinates, considered in this chapter, are very difficult to solve by means of rectangular coordinates. When one is familiar with various types of coordinates, he employs the kind best adapted to his purpose.

32. Polar coordinates

Evidently a person could be directed to a place by being told to go a certain **distance** in a certain **direction.** For example,

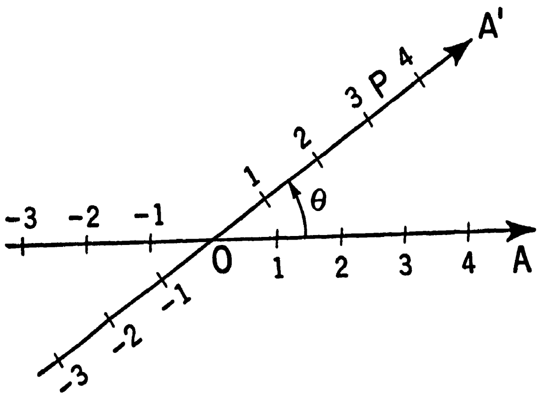


FIG. 1.

go 10 miles northwest. The polar coordinates of a point are distance and direction, the direction being specified by an angle.

Let OA (see Figure 1) be a line fixed in a plane and numbered like the x-axis with origin O as in §3, the direction from O to A

being considered positive.* *The point* O *is called the* **pole** *and the line* OA *the* **polar axis.** Rotate a line initially coinciding with OA, and numbered and directed the same as OA, about O through an angle θ, generally but not necessarily counterclockwise, to position OA'. Then the number ρ and the angle θ thus associated with a point P on OA' are its polar coordinates. A point with coordinates ρ and θ is designated by the symbol (ρ,θ). The point $(0,\theta)$ is the pole O whatever the angle θ may be. The vector $\overrightarrow{OP}$ is called the **radius vector** of P.

Figure 2(a) shows the coordinates of a number of points and Figure 2(b) shows the same points as Figure 2(a), with different

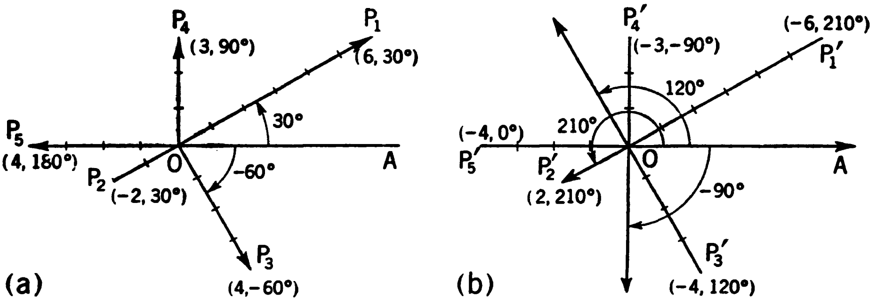


FIG. 2.

associated number pairs. To each number pair there is associated only one point, but to each point there are associated infinitely many number pairs. Evidently, to any point (ρ,θ) are associated the number pairs

$$(\rho,\theta + 2k\ 180°) \text{ and } (-\rho,\theta + (2k + 1)\ 180°), \tag{1}$$

where k is an integer.

Exercises

1. Plot: $P_1(6,30°)$, $P_2(10,120°)$, $P_3(12\sqrt{3},240°)$, $P_4(10\sqrt{2},-45°)$, $P_5(5,300°)$, $P_6(5,0°)$, $P_7(6,90°)$.

2. Express each of the points of Exercise 1 in two ways besides the one given, using for the purpose the least positive angle and the least negative angle available.

3. Plot the points: $(0,0°)$, $(0,90°)$, $(0,\theta)$.

* The line OA is here considered as a complete line extending indefinitely in both directions.

4. Using only non-negative angles less than 360°, how many sets of polar coordinates can be found for each point except the pole?

5. Plot the points (5,0°), (5,30°), (5,90°), (5,180°), (5,270°). What is the locus of points for which: (a) $\rho = 5$? (b) $\rho = a$?

6. Plot the points (1,30°), (2,30°), (5,30°), (−1,30°), (−2,30°), (−5,30°). What is the locus of points for which:

(a) $\theta = 30°$? (b) $\theta = \alpha$?

7. Plot the points given by the following table of values and connect them with a smooth curve:

θ	0°	30°	60°	90°	135°	180°	225°	270°	315°
$\rho = 2 \sin \theta$	0	1	1.73	2	1.41	0	−1.41	−2	−1.41

The table is computed from $\rho = 2 \sin \theta$.

8. Make a table of values for $\rho = 2 \cos \theta$ similar to the one in Exercise 7. Plot the corresponding points and connect them with a smooth curve. They should all lie on a circle.

9. Replace $2 \cos \theta$ in Exercise 8 by $4 \sin \theta$ and solve the resulting problem.

33. Relation between polar coordinates and rectangular coordinates

Figure 3 represents a pole O, polar axis OA, and rectangular axes with origin at O, x-axis along the polar axis, and the positive y-axis making an angle of 90° with line OA. By the definition of the trigonometric functions, we have, from Figure 3, $x/\rho = \cos \theta$, $y/\rho = \sin \theta$, or

$$x = \rho \cos \theta, \quad y = \rho \sin \theta, \tag{2}$$

and from these we easily obtain

$$\rho = \pm\sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x). \tag{3}$$

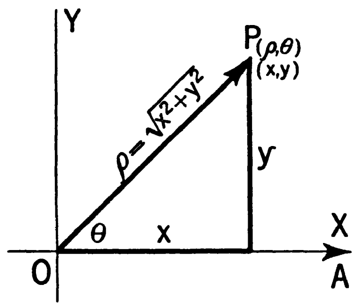


FIG. 3.

The sign + or − to be used and the corresponding angle θ to be used depend upon various conditions; in general, when the positive sign is adopted for ρ, θ is an angle satisfying

$$\cos \theta = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin \theta = \frac{y}{\sqrt{x^2 + y^2}}. \tag{4}$$

Observe that the relations (2), (3), and (4) are easily read from Figure 3.

Example 1. Express point (6,150°) in rectangular coordinates and the point (−4,−2) in polar coordinates.

Solution. Using formulas (2), we have:

$$x = 6\cos 150° = 6\left(-\frac{\sqrt{3}}{2}\right) = -3\sqrt{3},$$

$$y = 6\sin 150° = 6\left(\tfrac{1}{2}\right) = 3.$$

Hence, (6,150°) is expressed by **($-3\sqrt{3}$,3)**. Figure 4 illustrates the situation.

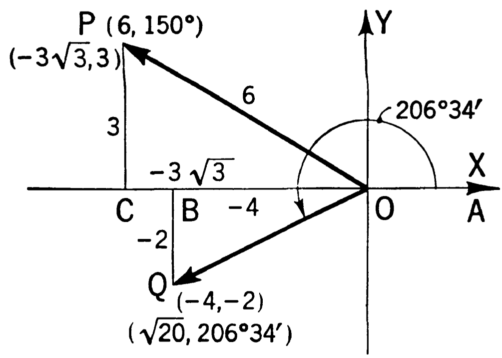


FIG. 4.

For the point (−4,−2), first get from Figure 4, tan (angle BOQ) $= \frac{1}{2}$. Therefore,

$$\tan(\text{angle } BOQ) = \tfrac{1}{2}, \text{ angle } BOQ = 26°\,34'.$$

Hence, the counterclockwise angle $AOQ = 180° + 26°\,34' = 206°\,34'$. Since the length of $OQ = \sqrt{4+16} = \sqrt{20}$, polar coordinates of Q are **($\sqrt{20}$,206° 34′)**.

Example 2. (a) Transform $x^2 + y^2 + 4x = 0$ to the corresponding polar form. (b) Transform $\rho^2 = 5\sin 2\theta$ to rectangular form.

Solution. (a) In $x^2 + y^2 + 4x = 0$ replace x and y by their values from (2) to obtain

$$(\rho\cos\theta)^2 + (\rho\sin\theta)^2 + 4\rho\cos\theta = 0,$$

or, dividing out ρ and replacing $\sin^2\theta + \cos^2\theta$ by 1,

$$\rho + 4\cos\theta = 0.$$

(b) To transform $\rho^2 = 5 \sin 2\theta$ to rectangular form, replace ρ by $\sqrt{x^2 + y^2}$ from (3) and $\sin \theta$ and $\cos \theta$ in $\sin 2\theta = 2 \sin \theta \cos \theta$ by their values from (4) to obtain

$$x^2 + y^2 = 10(y/\sqrt{x^2 + y^2})(x/\sqrt{x^2 + y^2}),$$

or, simplified,

$$\boldsymbol{(x^2 + y^2)^2 = 10xy.}$$

Exercises

1. Find the rectangular coordinates of:

(a) $(5\sqrt{2}, 45°)$. (d) $(8, 315°)$. (g) $(-3, 270°)$.
(b) $(10, 120°)$. (e) $(-3, 90°)$. (h) $(0, 68°)$.
(c) $(-6, 30°)$. (f) $(4, 180°)$. (i) $(-3, 0°)$.

2. Find polar coordinates of the points:

(a) $(3,3)$. (d) $(4\sqrt{3}, -4)$. (g) $(0,8)$.
(b) $(-4, 4\sqrt{3})$. (e) $(6,0)$. (h) $(-12,0)$.
(c) $(-6,-8)$. (f) $(-5,0)$. (i) $(0,0)$.

Replace x and y in each of the equations numbered 3 to 14 by their values from (2), thus obtaining polar equations of the loci represented by the rectangular equations:

3. $x = a$.
4. $y = -b$.
5. $x^2 + y^2 = a^2$.
6. $xy = a^2$.
7. $x^2 + y^2 - 6x = 0$.
8. $x^2 + y^2 + 4y = 0$.
9. $x^2 + y^2 = 6x + 8y$.
10. $y^2 = 6x$.
11. $x^2 + 4y^2 = 4$.
12. $x^2 - 4y^2 = 4$.
13. $(x^2 + y^2)^2 = x^2 - y^2$.
14. $x^2 + y^2 = \tan^{-1}(y/x)$.

Using equations (3) and (4) to express ρ and θ in terms of x and y, find in rectangular coordinates the equations of loci represented by Equations 15 to 26:

15. $\rho = a$.
16. $\rho \cos \theta = a$.
17. $\rho \sin \theta = b$.
18. $\rho^2 \sin \theta \cos \theta = a^2$.
19. $\rho = a \cos \theta$.
20. $\rho = a \sin \theta$.
21. $a\rho \cos \theta + b\rho \sin \theta = c$.
22. $\rho = a \sin \theta + b \cos \theta$.
23. $\rho = \sin 2\theta$.
24. $\rho = \cos 2\theta$.
25. $\rho^2 = a^2 \cos 2\theta$.
26. $\rho = a\theta$.

34. Plotting curves in polar coordinates

The graph of an equation in ρ and θ is a curve containing all points having a set of polar coordinates which satisfies the equation, and no others.

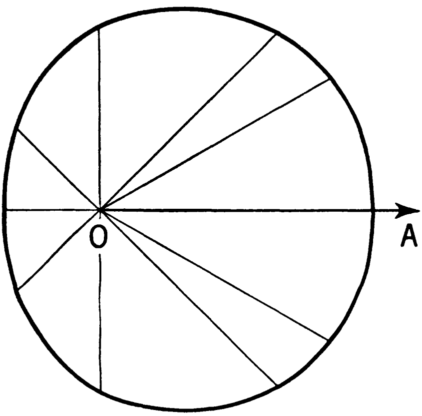


FIG. 5.

To plot a curve represented by an equation in ρ and θ, make a table containing associated values of ρ and θ from the given equation, plot the corresponding points, and connect them by a smooth curve. Consider, for example,

$$\rho = 2 + \cos\theta.$$

Assign values to θ, compute the corresponding values of ρ, and obtain the following table:

θ	0°	30°	45°	90°	135°	180°	225°	270°	315°	330°
ρ	3	2.9	2.7	2	1.3	1	1.3	2	2.7	2.9

Figure 5 shows a graph made by plotting the points corresponding to the tabular values and connecting them by a smooth curve.

Exercises

Plot the following curves, using intervals of 30° when applicable.

1. $\rho = 2 + \sin\theta$.

2. $\rho = 1 + \sin\theta$.

3. $\rho = 3\sin\theta$.

4. $\rho = 6\cos\theta$.

5. $\rho = 1 + \cos\theta$.

6. $\rho = 1 - \cos\theta$.

7. $\rho = \cos^2\theta$.

8. $\rho = a$.

9. $\theta = 90°$.

10. $\theta = 120°$.

11. $\rho = \sec^2 \frac{1}{2}\theta$.

12. $\rho = \frac{1}{60}\theta$, θ in degrees.

13. $\rho = \dfrac{1}{1 - \sin\theta}$.

14. $\rho = \dfrac{1}{1 + \cos\theta}$.

15. $(\rho - a)(\rho - b) = 0$.

16. $(\rho - a)(\rho - a\sin\theta) = 0$.

35. Sketching aids

When the rectangular equation corresponding to a given polar equation is easily plotted, it may well be obtained and used. For example, to plot $\rho = a \sec \theta$, write

$$\rho \cos \theta = a, \ \therefore x = a, \tag{5}$$

and plot the line $x = a$ shown in Figure 6. Similarly, to plot $\rho = a \csc \theta$, write

$$\rho \sin \theta = a, y = a, \tag{6}$$

and plot $y = a$ directly. Again, to sketch $\rho^2 \cos^2 \theta + 4\rho^2 \sin^2 \theta = 4$, obtain the corresponding rectangular equation:

$$x^2 + 4y^2 = 4$$

and sketch the corresponding ellipse.

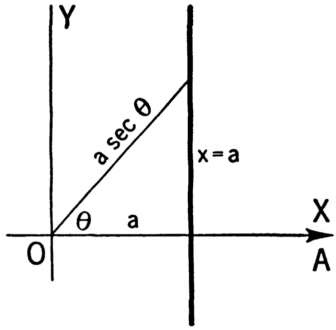


FIG. 6.

A very important method in sketching $\rho = f(\theta)$ consists in setting ρ equal to zero, solving the result for θ, and then using information deduced from the following theorem.

THEOREM. *If $\theta_1, \theta_2, \cdots$ are real roots of $f(\theta) = 0$, the curve $\rho = f(\theta)$ generally has a branch* tangent to the line $\theta = \theta_1$ at the pole, another branch tangent to $\theta = \theta_2$ at the pole, and so on.*†

Figure 7 shows two branches of a curve passing through the pole O tangent to the lines $\theta = \theta_1$ and $\theta = \theta_2$ and a typical corresponding loop. A few illustrations will show the usefulness of the theorem. Consider the equation

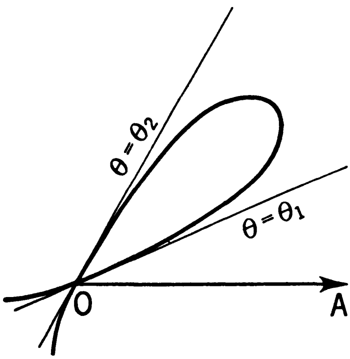


FIG. 7.

$$\rho = 2 \sin 3\theta. \tag{7}$$

Setting ρ equal to zero, we get

$$2 \sin 3\theta = 0, \ \therefore 3\theta = 0°, 180°, 360°, 540°, \cdots, \tag{8}$$

and, therefore,

$$\theta = 0°, 60°, 120°, 180°, \cdots. \tag{9}$$

This shows us that the curve passes through the origin tangent to

$$\theta = 0°, \theta = 60°, \theta = 120°, \cdots.$$

---

* "Branch" is a technical term. Here it will suffice to think of it as a certain portion of a curve.

† This theorem is proved in a more advanced subject called *calculus*.

Now from (7) compute the value pairs of the following table:

θ	0°	30°	60°	90°	120°	150°	180°
ρ	0	2	0	-2	0	2	0

The use of values of θ greater than 180° would give rise to no additional points.

Figure 8 shows the sketch of (7).* A closer approximation to the true shape would require the plotting of more points. The barbs

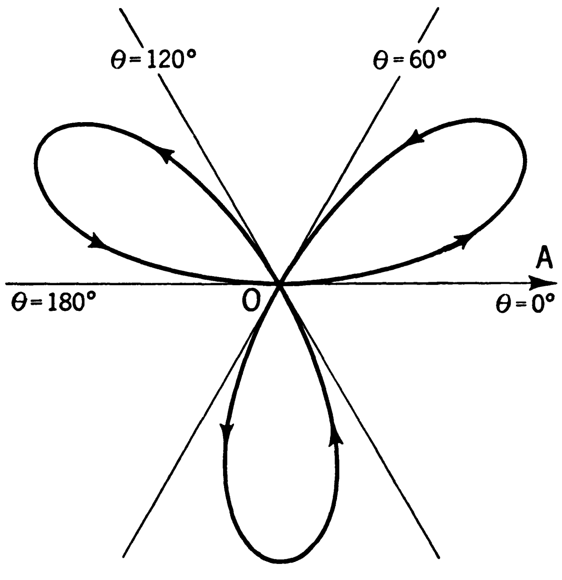


FIG. 8. The Three-Leaved Rose.

on the curve indicate the motion of a point describing the curve in the direction of increasing values of θ, tracing out one loop after the other continuously, and passing through the pole in the directions defined by (9).

Consider the curve, called the **cardioid,** represented by

$$\rho = a(1 - \cos\theta). \tag{10}$$

* The curve of Figure 8 illustrates the so-called *rose-leaved curves* defined by equations of the type $\rho = a \sin n\theta$ and $\rho = a \cos n\theta$. Every loop, associated with an equation, has the same size and shape.

Figure 9 shows the graph made by using the following table constructed from (10):

θ	$0°$	$\pm 45°$	$\pm 90°$	$\pm 135°$	$180°$
ρ	0	$0.29a$	a	$1.71a$	$2a$

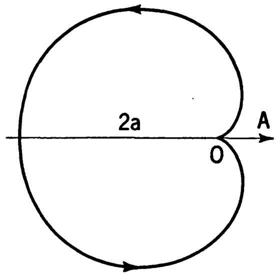


FIG. 9. The Cardioid $\rho = a(1 - \cos\theta)$.

Setting ρ equal to zero in (10), we obtain

$$a(1 - \cos\theta) = 0, \theta = 0° \text{ or } 360°.$$

Note that the sharp point at the pole would be predicted from the fact that $\theta = 0°$ and $\theta = 360°$ are lines tangent to the curve.

As another example take the curve, called the **limaçon,** represented by

$$\rho = 3(1 - 2\cos\theta). \tag{11}$$

Setting ρ equal to zero, we obtain

$$3(1 - 2\cos\theta) = 0, \cos\theta = \tfrac{1}{2}, \theta = -60°, 60°, 300°, \cdots \tag{12}$$

The following table of values was computed from (11):

θ	$\mp 60°$	$\mp 30°$	$0°$	$\pm 60°$	$\pm 90°$	$\pm 135°$	$180°$
ρ	0	-2.2	-3	0	3	7.24	9

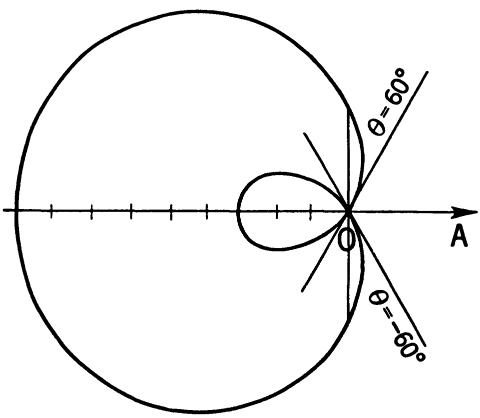


FIG. 10. The Limaçon.

Plotting the points corresponding to the table of values, and using the fact that $\theta = \pm 60°$ are tangent lines at the pole, we sketch the curve shown in Figure 10.

36. Symmetry

Consideration of symmetry often helps in sketching a curve. From Figure 11 it appears that

(ρ,θ) and $(\rho,-\theta)$ are symmetric to the polar axis. Hence, *an equation which is unchanged or changed to an equivalent* one when θ is replaced by $-\theta$ has a graph symmetric to the polar axis.* Since $\cos(-\theta) = \cos\theta$, we deduce that the graphs of Equations (10) and (11) are symmetric to the polar axis, and inspection of Figures 9 and 10 verifies this conclusion.

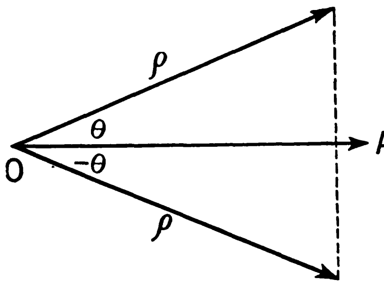


FIG. 11.

By using like arguments and the coordinates of points P', P'', and P''' of Figure 12, symmetric to P in various ways, we deduce the facts indicated in the table below. In this table the symbol $A \to B$ means *if when* A *is replaced by* B *in an equation, it is unchanged or changed to an equivalent one.*

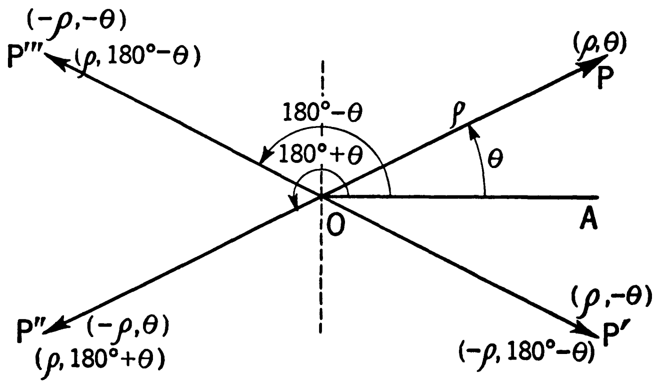


FIG. 12.

$$\left.\begin{array}{ll}\theta \to (-\theta) & \text{or} \\ \theta \to (180^\circ - \theta) \text{ and } \rho \to (-\rho)\end{array}\right\} \text{symmetry to polar axis} \quad (13)$$

---

* Two equations are equivalent if one may be obtained from the other by multiplying it through by 1 or -1. More generally, two polar equations are equivalent if they represent the same graph. Thus, $\rho = 1$ and $\rho = -1$ are equivalent.

$$\left.\begin{array}{l}\theta \to (180^\circ - \theta) \text{ or} \\ \theta \to (-\theta) \text{ and } \rho \to (-\rho)\end{array}\right\} \text{symmetry to } \theta = 90^\circ \qquad (14)$$

$$\left.\begin{array}{l}\rho \to (-\rho) \qquad \text{or} \\ \theta \to (180^\circ + \theta)\end{array}\right\} \text{symmetry to pole} \qquad (15)$$

Either one of the two conditions of (13), (14), or (15) is sufficient to imply the corresponding symmetry.*

The student should make no attempt to memorize this table; use of the basic principle will suggest the rule to be applied.

Example. Test for symmetry:

(a) $\rho = 2 \sin 3\theta$. (b) $\rho^2 = a \sin 2\theta$. (c) $\rho^2 = a \cos 2\theta$.

Solution. (a) Use the second test of (14). Since $-\rho = 2 \sin (-3\theta)$ is equivalent to $\rho = 2 \sin 3\theta$, the graph is symmetric to $\theta = 90^\circ$. (See Figure 8.) The first test of (14) gives the same result, since $2 \sin 3(180^\circ - \theta) = 2 \sin (540^\circ - 3\theta) = 2 \sin 3\theta$.

(b) Use the first test of (15). Since $\rho^2 = (-\rho)^2$, the graph of $\rho^2 = a \sin 2\theta$ is symmetric to the pole. (See Figure 13.)

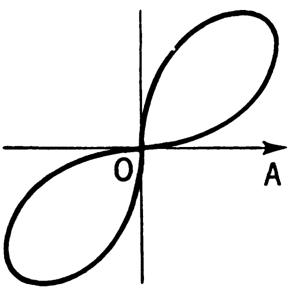


FIG. 13. The Lemniscate $\rho^2 = a^2 \sin 2\theta$.

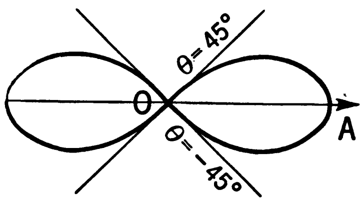


FIG. 14. The Lemniscate $\rho^2 = a^2 \cos 2\theta$.

(c) Each of the tests (13), (14), and (15) indicates the existence of the corresponding symmetry. Hence, $\rho^2 = a \cos 2\theta$ is symmetric to the polar axis, to $\theta = 90^\circ$, and to the pole. (See Figure 14.)

---

* Each line of statements (13), (14), and (15) suggests a substitution. The substitution suggested by any line, in an equation, gives the equation of a graph symmetric to that of the original equation with respect to the corresponding line or point. If one of a pair of conditions (13), (14), or (15) holds for a given equation, then the substitution suggested by the other of the pair in that equation gives an equation equivalent to the original one. Thus, the first condition of (13) is satisfied for $\rho = 1 + \cos \theta$; then $-\rho = 1 + \cos (180^\circ - \theta)$, or $\rho = -1 + \cos \theta$ has the same graph as $\rho = 1 + \cos \theta$.

Exercises

1. Express in rectangular coordinates and graph:

(a) $\rho = a \sec \theta$.
(b) $\rho \cos \theta = -b$.
(c) $\rho = a \csc \theta$.
(d) $\rho \sin \theta = -b$.

For each of the equations numbered 2 to 7, find equations of the tangent lines at the pole.

2. $\rho = a \cos 2\theta$.

3. $\rho = a \sin \theta$.

4. $\rho = a(1 - \sin \theta)$.

5. $\rho = a(1 + \cos \theta)$.

6. $\rho = a(1 - 2 \sin \theta)$.

7. $\rho = a(\sin \theta - \cos \theta)$.

Applying tests (13), (14), and (15) for symmetry, test for symmetry the graphs of the equations numbered 8 to 13:

8. $\rho = a \cos 3\theta$.

9. $\rho = a \sin \theta$.

10. $\rho^2 = \sin \theta + \cos \theta$.

11. $\rho = a \sin 2\theta + b \sin 4\theta$.

12. $\rho = a \tan \theta$.

13. $\rho = a \cos^4 \theta$.

For each of the equations numbered 14 to 17, find the positive values of θ less than 360° for which ρ is imaginary:

14. $\rho^2 = 2 \sin \theta$.

15. $\rho^2 = 4 \sin 2\theta$.

16. $\rho^2 = 4 \cos 2\theta$.

17. $\rho^2 + 2\rho = 2 \sin \theta$.

Sketch the graphs of the equations numbered 18 to 39:

18. $\rho = 3 \sin \theta$.

19. $(\rho - 3 \sin \theta)(\rho - 3 \cos \theta) = 0$.

20. $\rho = a \cos 2\theta$.

21. $\rho = a \sin 2\theta$.

22. $\rho = a(1 + \cos \theta)$.

23. $\rho = a(1 + \sin \theta)$.

24. $\rho = a(1 - \sin \theta)$.

25. $\rho = a \cos 3\theta$.

26. $\rho = a(1 - 2 \cos \theta)$.

27. $\rho = a(1 + 2 \sin \theta)$.

28. $\rho^2 = 4 \sin \theta$.

29. $\rho = a \sin^2 \theta$.

30. $\rho^2 + \rho - 2 = 0$.

31. $\rho^2 = 4 \rho + 21$.

32. $(\rho - a)(\rho - a \sin \theta) = 0$.

33. $(\rho - 2a \csc \theta)(\rho - 2a \sin \theta) = 0.$

34. $(\rho - a \sec \theta)[\rho - a(1 - \cos \theta)] = 0.$

35. $\rho^2 = a^2 \sin^2 \theta.$

36. $\rho = a \sec^2 \frac{1}{2}\theta.$

37. $\rho = a \csc^2 \frac{1}{2}\theta.$

★**38.** $\rho = \sin \frac{1}{3}\theta.$

★**39.** $\rho = \sin \frac{3}{2}\theta.$

Using the idea expressed in the footnote on page 97, show that the members of each pair of equations numbered 40 to 45 are equivalent:

40. $\rho^2 = \sin \theta,\ \rho^2 = -\sin \theta.$

41. $\rho = 1 + \sin \theta,\ \rho = -1 + \sin \theta.$

42. $\rho^2 = \cos \frac{1}{2}\theta,\ \rho^2 = -\sin \frac{1}{2}\theta.$

43. $\rho = a \sin^3 \theta + b \sin^2 \theta,\ \rho = a \sin^3 \theta - b \sin^2 \theta.$

44. $\rho = a \cos^3 \theta + b \cos^2 \theta,\ \rho = a \cos^3 \theta - b \cos^2 \theta.$

45. $\rho = \tan \theta,\ \rho = -\tan \theta.$

37. Rotation of polar axis

Figure 15 shows polar axis OA' making an angle α with polar axis OA, and point P in their plane with coordinates ρ,θ with respect to O as pole and OA as axis and ρ',θ' with respect to O as pole and OA' as axis. From the figure we read

$$\rho = \rho',\ \theta = \theta' + \alpha. \tag{16}$$

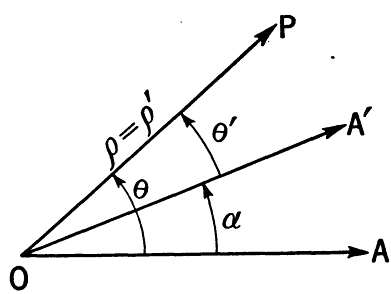


FIG. 15.

Hence, *to obtain from a given equation a new one representing the same graph referred to the same pole but a new polar axis making an angle α with the original one, replace ρ by ρ' and θ by $\theta' + \alpha$ in the given equation.*

Example. Sketch: (I) $\rho = 2 \cos (\theta - 30°)$.

(II) $\rho = a \csc (\theta + 60°)$.

Solution. (I) Replacing ρ by ρ' and θ by $\theta' + 30°$ in $\rho = 2 \cos (\theta - 30°)$, we obtain

$$\rho' = 2 \cos \theta'. \tag{a}$$

Figure 16 shows the two axes and the graph plotted from Equation (a).

(II) Replacing ρ by ρ' and θ by $\theta' + (-60°)$ in $\rho = a \csc (\theta + 60°)$, we obtain

$$\rho' = a \csc \theta', \text{ or } \rho' \sin \theta' = a. \tag{b}$$

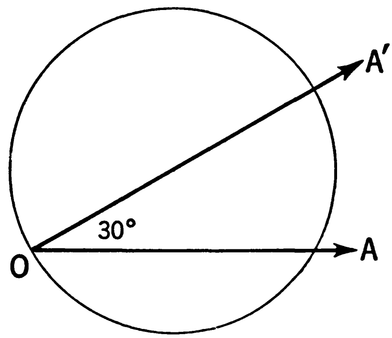


FIG. 16.

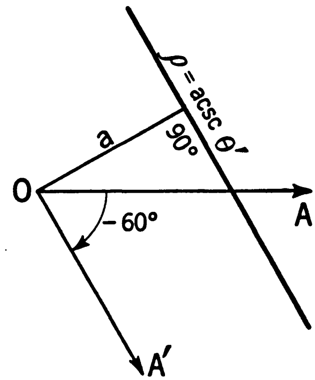


FIG. 17.

Figure 17 shows the two axes and the graph plotted from Equation (b) with respect to OA' as polar axis and O as pole.

Exercises

Transform each of the equations numbered 1 to 12 to a simple form by using (16). Draw both axes and plot the result relative to the original pole O and the new polar axis OA':

1. $\rho = 2 \sin (\theta - 60°)$.
2. $\rho = 3 \cos (\theta + 60°)$.
3. $\rho = 5 \csc (\theta + 135°)$.
4. $\rho = 4 \sec (\theta - 120°)$.
5. $\rho + a \sec (\theta + 315°) = 0$.
6. $\rho + 2a \csc (\theta - 120°) = 0$.
7. $\rho = a[1 - \cos (\theta - 30°)]$.
8. $\rho = a[1 + \sin (\theta + 60°)]$.
9. $\rho = a[1 + 2 \cos (\theta - 120°)]$.
10. $\rho = a[1 - 2 \sin (\theta + 150°)]$.
11. $[\rho - a \sin (\theta - 60°)][\rho - a \csc (\theta + 60°)] = 0$.
12. $[\rho^2 - a \sin (2\theta - 60°)][\rho^2 - 2a \cos (2\theta + 30°)] = 0$.

38. Equations of loci in polar form

To find the equation of a locus of points satisfying given conditions, let (ρ,θ) represent any point on the locus, draw a figure showing essential relations, express the given conditions in the

form of an equation in ρ and θ, and simplify the result. A few examples will illustrate the method.

Example 1. Find the equation of the locus of a point distant $2a$ from the point $(a,0)$.

Solution. Figure 18 shows the point $C(a,0)$ and any point $P(\rho,\theta)$ on the locus. Applying the law of cosines* to the triangle OCP, we get

$$4a^2 = \rho^2 + a^2 - 2a\rho \cos \theta,$$

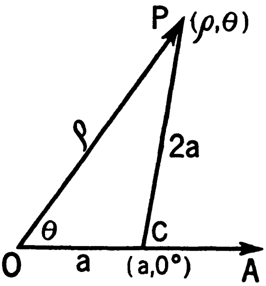


FIG. 18.

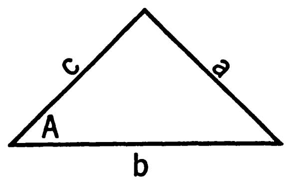


FIG. 19.

or, simplified,

$$\rho^2 - 2a\rho \cos \theta = 3a^2.$$

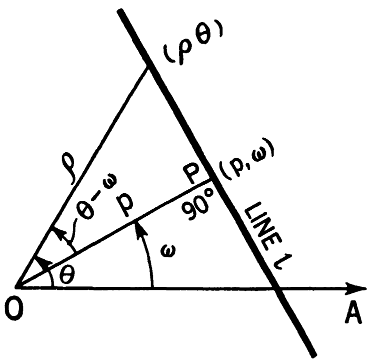


FIG. 20.

Example 2. Find the equation of a line l through point (p,ω) and perpendicular to the line connecting pole O and (p,ω) (see Figure 20).

Solution. Figure 20 shows the required line l through (p,ω) perpendicular to OP and (ρ,θ) any point on line l. From the figure we read

$$p/\rho = \cos (\theta - \omega), \quad \text{or}$$

$$\rho = p \sec (\theta - \omega). \tag{17}$$

Observe that this is a general equation for a straight line.†

* The law for the general triangle of Figure 19 states that

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

† It is interesting to observe that we obtain from (17)

$$\rho \cos (\theta - \omega) = p,$$
$$(\rho \cos \theta) \cos \omega + (\rho \sin \theta) \sin \omega = p,$$
$$x \cos \omega + y \sin \omega = p,$$

the normal form of the equation of a straight line.

Example 3. Through a fixed point O a line is drawn meeting a fixed line l in B, and points P are found on OB or its prolongation distant m from B. Find the locus of points P thus obtainable.

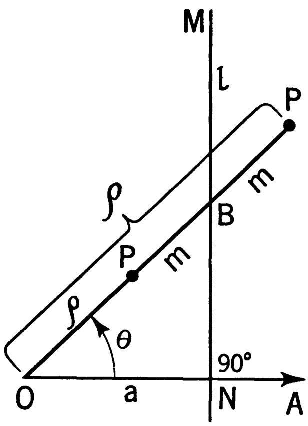


FIG. 21.

Solution. Take the fixed point O as pole and a perpendicular from O to the fixed line MN as polar axis. A general position of point P is indicated in Figure 21. Represent the distance from O to line MN by a. Then, since $OP = OB + BP$ and $OB = a \sec \theta$,

$$\rho = a \sec \theta \pm m.$$

It turns out that the equation represents the complete locus when either the sign $+$ or the sign $-$ is used. Figure 22 represents the graph when $m > a$, Figure 23 when $m < a$, and Figure 24 when $m = a$. The graphs are called **conchoids.**

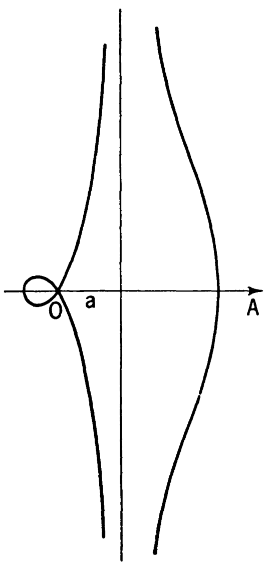


FIG. 22.

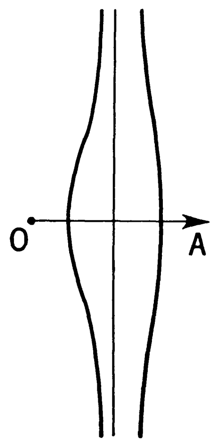


FIG. 23.

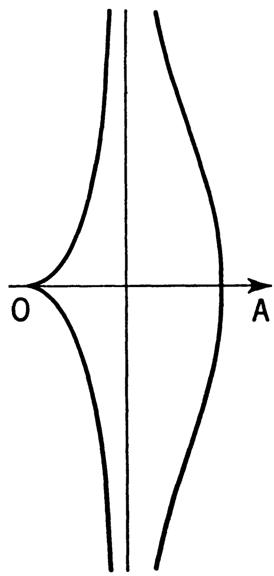


FIG. 24.

Exercises

Find equations of the loci described in the exercises numbered 1 to 13:

1. Points distant a from the pole.

2. Points distant a from point $(a,90°)$.

3. Points distant a from point $(a,180°)$.

4. Points on a line perpendicular to the polar axis and passing through $(a,0°)$.

5. Points on a line parallel to the polar axis and passing through $(b,270°)$.

6. Points on a circle with center $(2a,0)$ and radius $2a$.

7. Points on a circle having x-intercept a and y-intercept b and passing through the pole. First find the equation in rectangular coordinates.

8. Points at the same distance from the pole as from the line $\rho = a \sec \theta$.

9. Points on a line which cuts $\theta = 150°$ at right angles in $(a,150°)$.

10. Points on a line which cuts $\theta = \alpha$ at right angles in (a,α).

11. Points twice as far from the pole as from the polar axis.

12. Points twice as far from the pole as from $\theta = 90°$.

13. Points (ρ',θ') obtained by adding a to each value of ρ on $\rho = a \sec \theta$ to obtain ρ' and α to θ to obtain θ'.

14. A line is drawn through a fixed point O on a circle of radius a and meets it again in Q. Two points are located, on the line, distant m from Q. Find the equation of the locus of points thus obtainable.

15. Through one end O of a fixed diameter of a circle of radius r is drawn a line meeting the circle in Q and the tangent to the circle at the other end of the diameter in R. Point P is located on line OQ distant QR from O. Find the locus of P.

★16. In Figures 25, 26, and 27 OB is a fixed diameter. Find the equation of the locus of P for each figure. Assume that P takes all positions on OR consistent with the equation $|\overrightarrow{OP}| = |\overrightarrow{QR}|$.

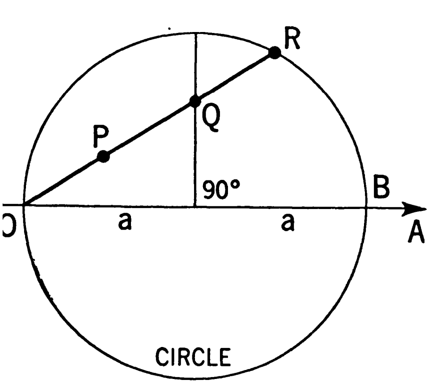


FIG. 25.

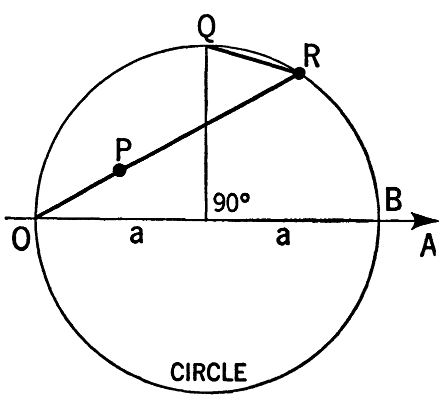


FIG. 26.

★**17.** Show that an equation of a line through (ρ_1,θ_1) and (ρ_2,θ_2) is $\rho[\rho_1 \sin(\theta_1 - \theta) + \rho_2 \sin(\theta - \theta_2)] = \rho_1\rho_2 \sin(\theta_1 - \theta_2)$.

Hint. Write the equation of the line in rectangular coordinates, transform to polar coordinates, and simplify.

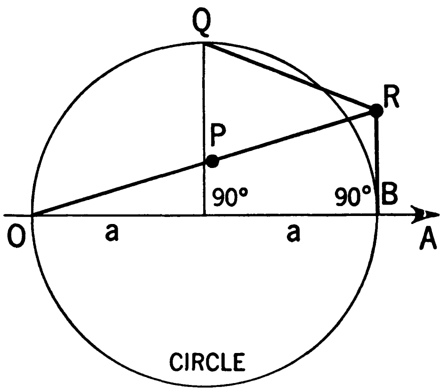


FIG. 27.

39. Intersections of graphs of polar equations

The simultaneous solutions of two equations in polar coordinates always represent points of intersection of their graphs, but there may be intersections corresponding to which the equations have no simultaneous solutions. For example,

$$\rho = a \sin \theta, \rho = a \cos \theta \tag{18}$$

intersect at the pole (see Figure 28), but for no value of θ are $a \sin \theta$ and $a \cos \theta$ both zero. As another illustration, consider

$$\rho = 1, \rho = \tan \theta. \tag{19}$$

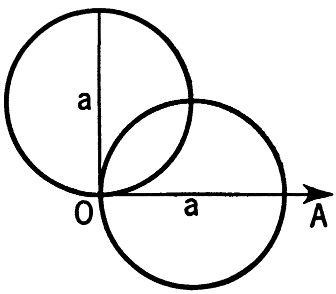


FIG. 28.

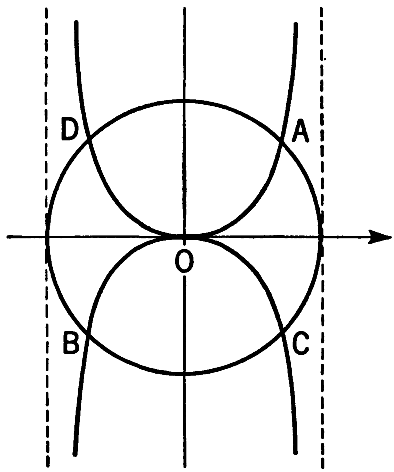


FIG. 29.

Figure 29 shows the graphs intersecting at A, B, C, and D. The simultaneous solutions of (19), coming from $\tan \theta = 1$, are

$$\rho = 1, \theta = \frac{\pi}{4} + k\,180°; \ (k \text{ an integer}). \tag{20}$$

These represent only the points A and B on the graph.

In obtaining the coordinates of C and D, consider that if (ρ_1,θ_1) satisfies the first of two polar equations, and if one of the other values from (1), §32

$$(\rho_1,\theta_1 + 2k\ 180°), \text{ or } (-\rho_1,\theta_1 + [2k + 1]\ 180°), \tag{21}$$

representing the same point, satisfies the second, then (ρ_1,θ_1) is a point of intersection.* Using the second set of values from (21) in the second equation, we see that solutions of

$$\rho = 1,\ -\rho = -1 = \tan\,[\theta + (2k + 1)\ 180°] = \tan\theta \tag{22}$$

represent intersections of the graphs of (19). From (22), we obtain

$$(1, 135°) \text{ and } (1, 315°)$$

as points of intersection† of the graphs of (19).

To find the intersections of the graphs of $\rho = f_1(\theta)$ *and* $\rho = f_2(\theta)$, *proceed as follows:*

(a) *Determine whether there is a value* θ_1 *for which* $(0,\theta_1)$ *lies on the graph of the first equation and a value* θ_2 *for which* $(0,\theta_2)$ *lies on the second. If so, the pole is a point of intersection.*

(b) *Find the simultaneous solutions of the following pairs of equations:*

$$\rho = f_1(\theta), \quad \rho = f_2(\theta), \tag{23}$$

$$\rho = f_1(\theta), \quad \rho = f_2(\theta + 2k\ 180°), \tag{24}$$

$$\rho = f_1(\theta), \ -\rho = f_2(\theta + [2k + 1]\ 180°). \tag{25}$$

In most simple cases, the points of intersection are found by investigating the pole and solving the given equations simultaneously.

---

* The student should draw the graphs of the given equations. They show the situation at the pole, enable one to visualize the equations, and suggest, from symmetry and other considerations, means of finding unusual solutions.

† $\rho = f(\theta)$, $\rho = f(\theta + k\ 360°)$, and $-\rho = f(\theta + [2k + 1]\ 180°)$ all represent the same graph. For, if any point (m,α) satisfies the first of these equations, then the same point $(m,\alpha - k\ 360°)$ satisfies the second, and the same point $(-m,\alpha - [2k + 1]180°)$ satisfies the third. This same argument shows that any point which satisfies one of the three equations will satisfy the others. Hence, any point on one graph must be on the others, and any point not on one graph cannot be on the others.

Exercises

Find the points of intersection of the graphs of the pairs of equations numbered 1 to 12:

1. $\rho = \frac{1}{2}$, $\rho = \cos\theta$.

2. $\rho = \frac{1}{2}$, $\rho = \sin\theta$.

3. $\rho = \csc\theta$, $\rho = 2\sin\theta$.

4. $\rho = a(1 - \sin\theta)$, $\rho = a(1 - \cos\theta)$.

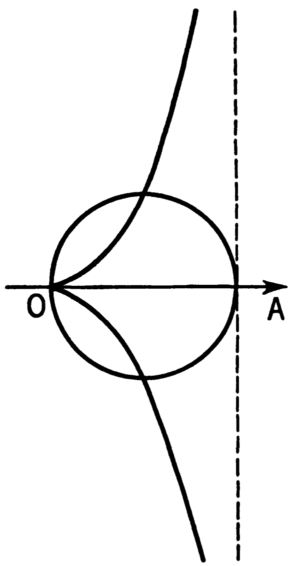


FIG. 30.

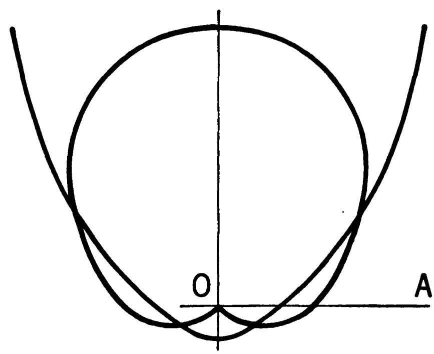


FIG. 31.

5. $\rho = 2\cos\theta$, $\rho\cos\theta = -\frac{1}{2}$.

6. $\rho\sin\theta = 2$, $\rho = 4$.

7. $\rho = 4\sin\theta$, $\rho\sin\theta = 3$.

8. $\theta = 90°$, $\rho = \cos 2\theta$.

9. $\rho = \sin\theta$, $\rho = \sin 2\theta$.

10. $\rho = \tan\theta$, $\rho = 2\sin\theta$.

11. $\rho = 2\tan\theta\sin\theta$, $\rho = 2\cos\theta$ (see Figure 30).

12. $\rho = 4(1 + \sin\theta)$, $\rho = 3/(1 - \sin\theta)$ (see Figure 31).

Find the points of intersection of the graphs of the sets of equations numbered 13 to 20. Expect to use Equations (24) and (25).

★13. $\rho = \sin 2\theta$, $\rho = \cos 2\theta$.

★14. $\rho = \tan 2\theta$, $\rho = 1$.

15. $\theta = 30°$, $\rho = 2$.

★16. $\rho = \cos\theta - 1$, $\rho = \cos 2\theta$.

★17. $\rho = 1 + \sin\theta$, $\rho = \cos 2\theta$.

★18. $\rho = 1 + \cos\theta$, $\rho = \cos 2\theta$.

19. $\rho^2 = 2\cos\theta$, $\rho = 1$.

★20. $\rho = \sin(\theta/2)$, $\rho = \frac{1}{2}$.

CHAPTER VII

Parametric Equations

40. An illustration of parametric equations

In the *parametric representation* of a curve, each of the variables x and y is expressed in terms of a third variable called a **parameter.** An illustration will bring out essential facts. If the x-axis is taken horizontal and the y-axis vertical, and if x and y are measured in feet and the time t in seconds, the equations

$$\begin{aligned} x &= 96t \\ y &= 96t - 16t^2 \end{aligned} \tag{1}$$

give approximately the position at time t of a heavy stone thrown, under certain conditions, from the origin at time $t = 0$. Assigning various values to t and computing the corresponding values of x and y from (1), we obtain the following table:

t	0	1	2	3	4	5	6
x	0	96	192	288	384	480	576
y	0	80	128	140	128	80	0

Plotting the points (x,y) from the table, without considering t, and connecting them by a smooth curve, we obtain (see Figure 1) the graph of Equation (1).

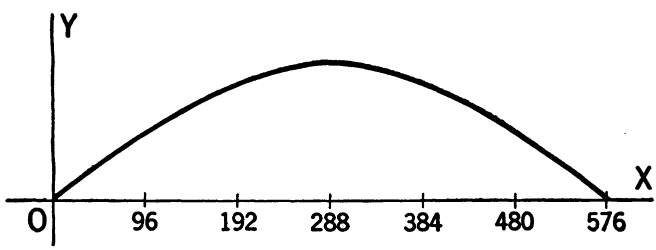


FIG. 1.

In general, parametric equations of a curve tell more than an equation in rectangular form of the same curve. For example, Equations (1) define the path of the stone and also tell when it reaches particular positions; that is, they describe a motion. Moreover, in many cases, a set of parametric equations can be found that is simpler than the corresponding rectangular equation.

Since the value of t in the first equation of (1) is equal to its value in the second, substitute $t = x/96$ from the first for t in the second to obtain

$$y = 96(x/96) - 16(x/96)^2, \tag{2}$$

the rectangular equation of the curve (1).

41. Parametric equations

Section **40** illustrates some important features of parametric representation of functions. General parametric equations may be represented by

$$x = f_1(t),\ y = f_2(t), \tag{3}$$

where t is a third variable called a *parameter.*

To plot a curve from parametric equations, assign values to t, *compute the corresponding values of* x *and* y, *plot the points* (x,y) *without reference to* t, *and connect them with a smooth curve.*

To obtain the equation in x *and* y *from parametric equations having the form* (3), *eliminate the parameter from the given equations.* For example, from

$$x = a + m \sin kt,\ y = b + n \cos kt \tag{4}$$

obtain $\sin kt = (x - a)/m$, $\cos kt = (y - b)/n$, and

$$\frac{(x-a)^2}{m^2} + \frac{(y-b)^2}{n^2} = \sin^2 kt + \cos^2 kt = 1. \tag{5}$$

From

$$x = a + m \tan kt,\ y = b + n \sec kt \tag{6}$$

obtain $\tan kt = (x - a)/m$, $\sec kt = (y - b)/n$, and

$$\frac{(y-b)^2}{n^2} - \frac{(x-a)^2}{m^2} = \sec^2 kt - \tan^2 kt = 1. \tag{7}$$

The eliminant may represent extraneous loci. For example, from

$$x = \cos^2 t,\ y = \sin^2 t \tag{8}$$

we obtain

$$x + y = \cos^2 t + \sin^2 t = 1. \tag{9}$$

Note that (8) represents only a small segment of the line (9), the segment in the first quadrant, since in (8) x and y are both positive and not greater than 1.

There are infinitely many ways of finding parametric equations of a graph from a given equation in x and y. In fact, *any function of* t, *with suitable restrictions, may be assigned to* x, *and* y *may then be found by substituting the function of* t *in the given equation and solving the result for* y. Equations are generally selected to obtain meaning, simplicity, or both. For example, to get parametric equations of $x - y = 1$, take $x = 2 + t/\sqrt{2}$, and obtain $2 + (t/\sqrt{2}) - y = 1$, or $y = 1 + t/\sqrt{2}$. Hence, the parametric equations are

$$x = 2 + t/\sqrt{2},\ y = 1 + t/\sqrt{2}. \tag{10}$$

From Figure 2 one deduces that t in (10) represents the directed distance from (2,1) to the point (x,y).

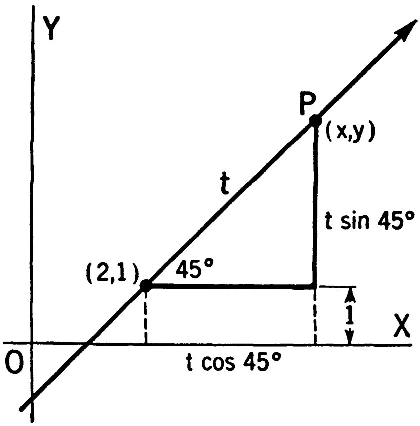


FIG. 2.

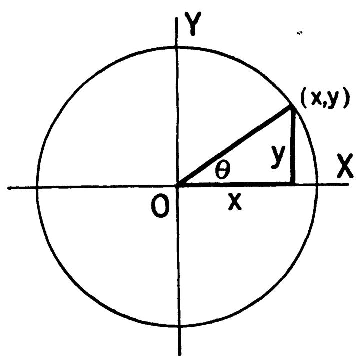


FIG. 3.

Consider the problem of finding simple parametric equations of a circle having radius a. Choosing the angle θ indicated in Figure 2 as parameter, we obtain from the figure

$$x = a\cos\theta,\ y = a\sin\theta \tag{11}$$

as parametric equations. If θ be replaced by kt, where t represents time, we obtain

$$x = a \cos kt, \; y = a \sin kt, \tag{12}$$

and these define a motion of point (x,y) at constant speed along the circle, as well as the circle.

Exercises

1. Sketch $x = t + 1$, $y = t^2$ for the range $t = -2$ to 2.

2. Sketch $x = t^2 + 1$, $y = 2t$ for the range $t = -2$ to 2.

3. Expressing ϕ in radians (1 radian $= (180/\pi)^\circ = 57.3^\circ$ approximately), sketch

$$x = \phi + \sin \phi, \; y = \phi - \sin \phi$$

for the range $\phi = 0$ to 2π radians.

4. Sketch $x = 3 \cos \phi$, $y = 4 \sin \phi$.

5. Sketch $x = 3 \tan \phi$, $y = 3 \sec \phi$ for the range $\phi = -60^\circ$ to 60°.

6. Sketch $x = \cos 2\phi$, $y = \cos \phi$.

From each set of equations numbered 7 to 17 eliminate the parameter to obtain a single corresponding equation in x and y:

7. $x = t + 2$, $y = t - 1$.

8. $x = t + 2$, $y = t^2$.

9. $x = kt$, $y = \pm\sqrt{1 - k^2t^2}$.

10. $x = t^2$, $y = t^3$.

11. $x = \sqrt{t}$, $y = t - 3$.

12. $x = 2 \sin 3t$, $y = 3 \cos 3t$.

13. $x = 3 + 2 \sin 3t$, $y = 2 - 3 \cos 3t$.

14. $x = a \cos^3 \phi$, $y = a \sin^3 \phi$.

Hint. $x^{\frac{2}{3}} = a^{\frac{2}{3}} \cos^2 \phi$.

15. $x = a \cos^4 \phi$, $y = a \sin^4 \phi$.

16. $x = 2 \sec 3\phi$, $y = 2 \tan 3\phi$.

17. $x = t + \dfrac{1}{t}$, $y = t - \dfrac{1}{t}$.

In the exercises numbered 18 to 21, find parametric equations of the curves represented, assuming that $y/x = t$.

18. $x^2 + y^2 = 2x$.

19. $x^2 - y^2 = 3x + y$.

20. $x^3 + y^3 = axy$.

21. $x^2 + xy + y^2 = 3x$.

Assuming the relations (2), (3), and (4), §**33**, find in rectangular form and in polar form the equations of the graphs determined by the pairs of equations numbered 22 to 26. Sketch the graphs:

22. $x = 1 + \cos t,\ y = 2 - \sin t.$

23. $x = 4 \sin t,\ y = 3 \sin t.$

24. $x = 1 + \tan t,\ y = \sec t.$

25. $x = 2 \sin^2 t \cos t,\ y = 2 \sin t \cos^2 t.$

Hint. $x^2 + y^2 = 4 \sin^2 t \cos^2 t,\ y/x = \cot t.$

26. A fly, starting at a fixed point O on a wire, crawls along the wire at 8 ft./min. while the wire rotates about O in a plane at 2 revolutions per minute. Using polar coordinates ρ and θ for the position of the fly, find ρ and θ in terms of time t. Sketch the path of the fly.

42. The involute of a circle

This and the following two articles consider some interesting curves having simple parametric equations. The radian (see

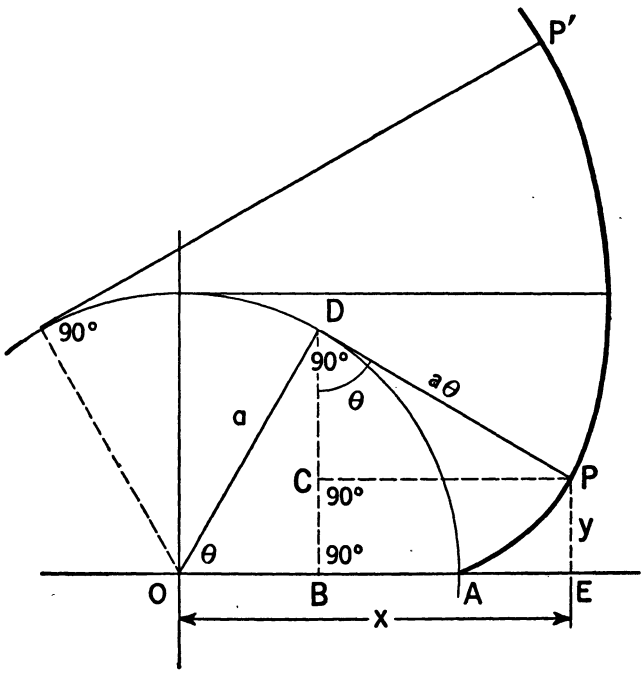


FIG. 4.

page 256) will be assumed as the unit of angular measurement. Also, frequent use will be made of the fact that the arc length s

of a circle is the product of the corresponding central angle θ in radians and the radius a; that is, $s = a\theta$.

An involute of a circle is the curve described in the plane of the circle by the end of a thread that is kept taut while being unwrapped from the circle. Imagine a thread wrapped around the circle of Figure 4 with one end initially at A. Curve APP' represents the path described by this end as the string is unwrapped, and P represents any position of the end. The length of arc AD is equal to the length DP, since the piece of string DP was in coincidence with arc AD at the start; hence,

$$DP = a\theta. \tag{13}$$

From the figure, we obtain for point P,

$$x = OE = OB + CP = a\cos\theta + a\theta\sin\theta$$

$$y = EP = BD - DC = a\sin\theta - a\theta\cos\theta.$$

Accordingly, the equations of the involute are

$$x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta). \tag{14}$$

These equations are fairly simple and the parameter θ is significant. The rectangular equation and the polar equation corresponding to (14) are rather complicated. Involutes of circles play a prominent role in the design of gear teeth.

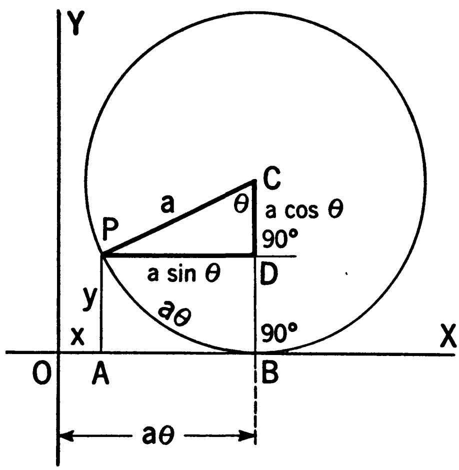


FIG. 5.

43. The cycloid

The path of a point on the rim of a wheel as it rolls along a straight line in a fixed plane containing the wheel is a **cycloid.** Figure 5 represents the circle C rolling along the x-axis. Assume that the point P, fixed to the rim, was at O when center C was on OY. Now, length of arc PB is equal to length OB, since

arc PB was rolled unit for unit against segment OB. Hence, using the symbols and relations indicated in Figure 5, we have

$$OB = a\theta \tag{15}$$

and

$$\begin{aligned} x &= OA = OB - PD = a\theta - a\sin\theta, \\ y &= AP = BD = BC - DC = a - a\cos\theta. \end{aligned} \tag{16}$$

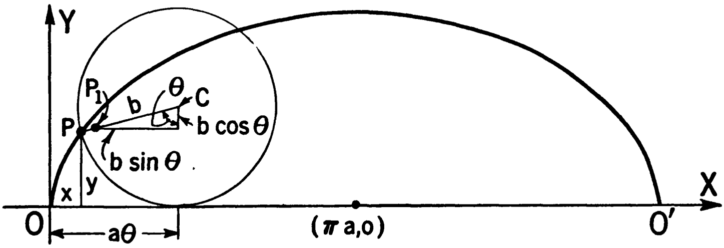


FIG. 6. The Cycloid.

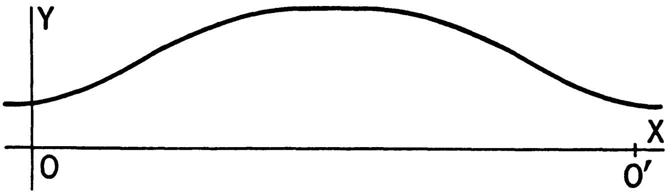


FIG. 7. The Curtate Cycloid.

Accordingly, the equations of the cycloid are

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta). \tag{17}$$

Figure 6 shows one arch of a cycloid.

The equations of the path of any point P_1 on the spoke CP of the wheel, or the spoke prolonged, are easily derived from Figure 6. They are

$$\begin{aligned} x &= a\theta - b\sin\theta, \\ y &= a - b\cos\theta. \end{aligned} \tag{18}$$

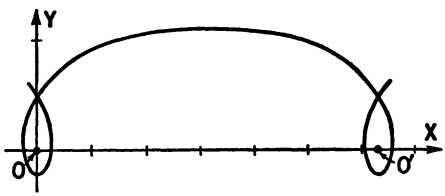


FIG. 8. The Prolate Cycloid.

If $b < a$, the curve (see Figure 7) defined by (18) is called a **curtate cycloid**; if $b > a$, the curve (see Figure 8) is called a **prolate cycloid.**

44. Epicycloids and hypocycloids

The plane locus of a point P fixed on a circle which rolls on the outside of a fixed circle is called an **epicycloid.** Figure 9 represents the fixed circle with center O and the moving circle with center C. Let P represent a general position of the point, fixed on the rolling circle C, which was in contact with A when

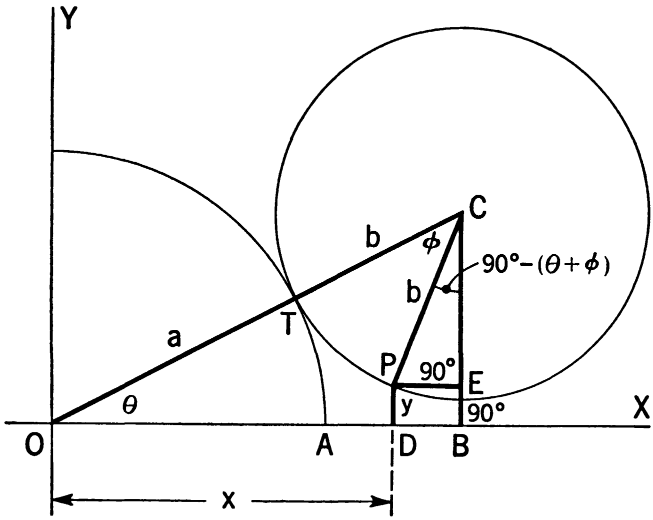


FIG. 9.

point C was on OX. Here arcs AT and PT are equal in length. Using the symbols and relations in Figure 9, we have

$$a\theta = b\phi, \tag{19}$$

$$\begin{aligned} x &= OB - DB = OB - PE = (a+b)\cos\theta - b\cos(\theta+\phi) \\ y &= DP = BC - EC = (a+b)\sin\theta - b\sin(\theta+\phi). \end{aligned} \tag{20}$$

Replacing ϕ by $a\theta/b$ from (19), we get as the equations of the epicycloid

$$\begin{aligned} x &= (a+b)\cos\theta - b\cos[(a+b)\theta/b] \\ y &= (a+b)\sin\theta - b\sin[(a+b)\theta/b]. \end{aligned} \tag{21}$$

If $a = b$, Equations (21) represent a **cardioid.** Its equations are

$$\begin{aligned} x &= 2a\cos\theta - a\cos 2\theta \\ y &= 2a\sin\theta - a\sin 2\theta. \end{aligned} \tag{22}$$

If in (21) we replace b by $-b$, the equations represent the locus of a point on the moving circle when it rolls on the inner side of the fixed circle. This locus is called a **hypocycloid**. Replacing b by $-\frac{1}{4}a$ in (21), we obtain the equations of the **hypocycloid of four cusps** shown in Figure 10. Its equations are

$$x = \tfrac{3}{4}a\cos\theta + \tfrac{1}{4}a\cos 3\theta, \quad y = \tfrac{3}{4}a\sin\theta - \tfrac{1}{4}a\sin 3\theta, \tag{23}$$

and these can be reduced to

$$x = a\cos^3\theta,\ y = a\sin^3\theta. \tag{24}$$

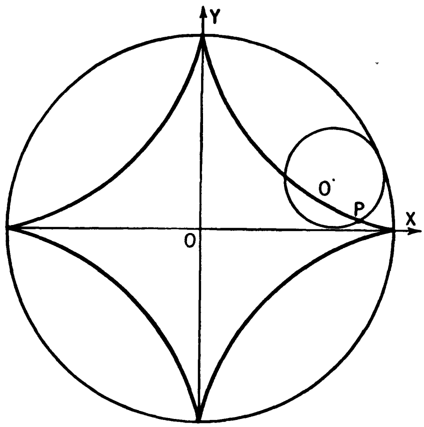


FIG. 10.

Exercises

1. Using (14), find the equations of the involute of a circle of radius 1 and sketch it.

2. Assuming that the inclination of the tangent at any point of the involute is the value of the parameter θ for the point, find the equation of the tangent when

(a) $\theta = \pi/3$; (b) $\theta = 2\pi/3$; (c) $\theta = \pi/2$.

3. The curve symmetric to the x-axis with the cyloid (17) is called the **brachistochrone**. Find its equations.

4. Show from (17) that the equation of the cycloid in rectangular form is

$$x = a\cos^{-1}\frac{a-y}{a} \mp \sqrt{2ay - y^2}.$$

5. Find by translation of axes the equations of the cyloid if the origin is taken at the highest point on it.

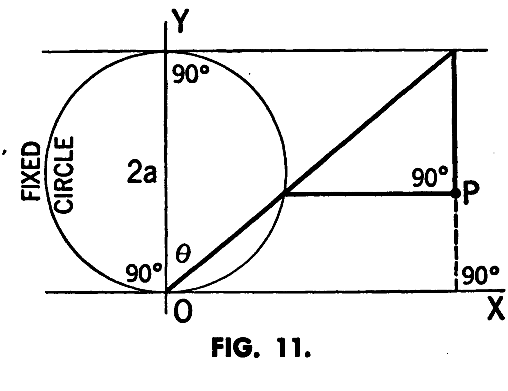


FIG. 11.

6. Take $b = \frac{1}{2}a$ in Equations (21) and sketch the curve defined by the result.

7. Find the general equations of the hypocycloid defined in the paragraph immediately following Equation (22).

8. Using the result from Exercise 7, find the equation of the hypocycloid when the radius of the moving circle is one-half that of the fixed circle. Sketch the locus.

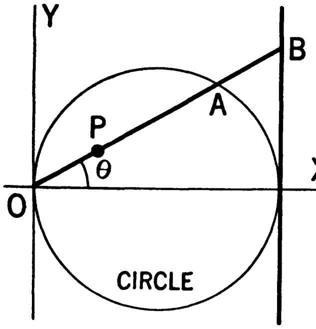


FIG. 12.

9. Find parametric equations of the locus of P defined for each value of θ by Figure 11. Sketch the locus and find its equation in rectangular form. The curve is called the *witch of Agnesi.*

10. For each value of θ the length of OP in Figure 12 is equal to that of AB. Using θ as parameter, find parametric equations of the locus of P. Also find its equations in rectangular form and in polar form. This locus is called the *cissoid of Diocles.*

11. In Figure 13, $BP = P'B = a$. Using θ as parameter, find parametric equations of the locus of P and of P'. In one set of equations replace θ by $180° + \theta$ to obtain the other set. Does this prove that either of the two sets of equations represents the locus of all points P and P'?

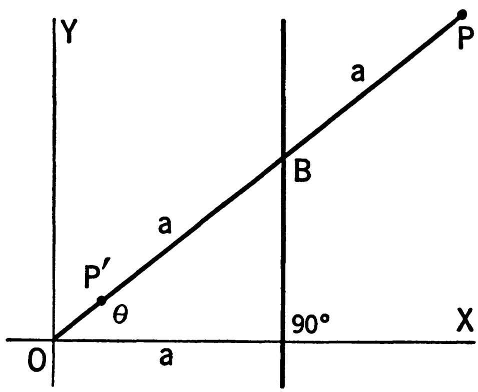


FIG. 13.

CHAPTER VIII

Conic Sections

45. Remarks

This chapter deals with the most important types of curves represented by the general equation of the second degree in two variables

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{1}$$

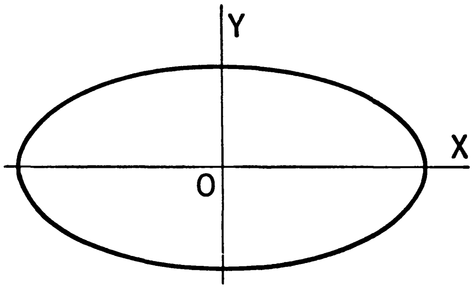


FIG. 1. Ellipse.

An equation of type (1) may represent an ellipse (see Figure 1), a parabola (see Figure 2), or a hyperbola (see Figure 3). Besides these, it may represent no real curve at all, a point, a straight line, or a pair of straight lines. The Greeks studied these curves as plane sections of a cone. In Figure 4, section $ABCD$ represents an ellipse, section EFG a parabola, and section $HIJKLM$ a hyperbola. Evidently, a plane section through vertex V might be the single point V, a single straight line, such as VK, or two straight lines, such as BV and DV. Figure 5 shows a fanciful

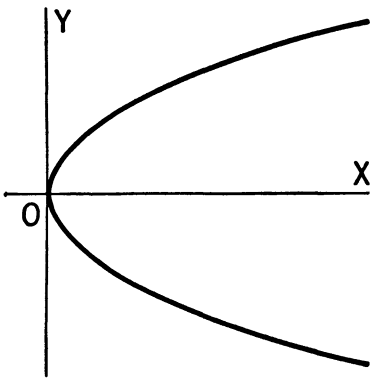


FIG. 2. Parabola.

picture of the young Isaac Newton obtaining on the back of a chair an elliptical section of a cone of light. In this chapter the ellipse, parabola, and hyperbola will be considered, and in the

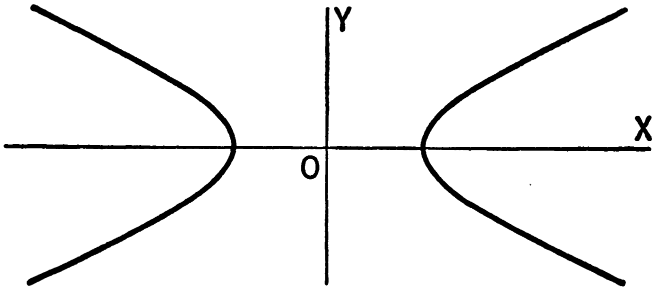


FIG. 3. Hyperbola.

next, the general equation of the second degree will be analyzed and shown to represent the types of loci mentioned above.

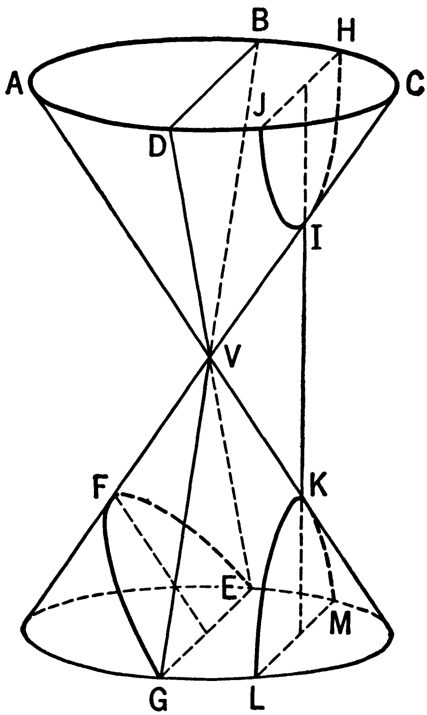
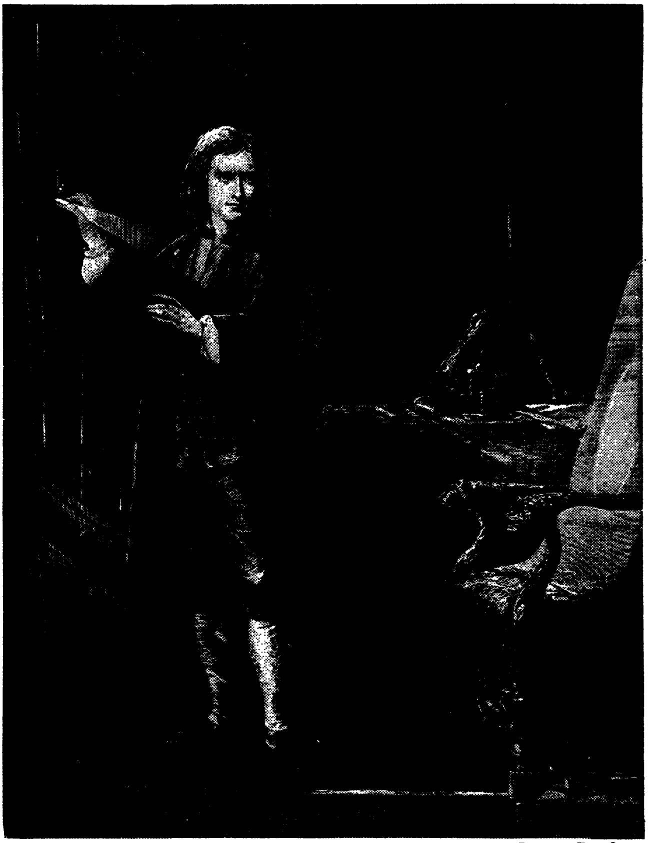


FIG. 4.



Brown Brothers.

FIG. 5. Sir Isaac Newton.

46. The conic section

DEFINITION. *A* **conic section** *is the locus of points so situated that the ratio of the distance of each from a fixed point to its distance from a fixed line not through the fixed point is a constant.*

The fixed point is called the **focus** of the conic, the fixed line is called its **directrix**, and the constant ratio, generally denoted by e, is called its **eccentricity**.

In Figure 6, F represents the focus, DD' the directrix. Take DD' as y-axis and the perpendicular to DD' through F as the x-axis. Call the origin O, let p be the directed* distance OF,

---

* p is a positive number or a negative number according as $\overrightarrow{OF}$ has the positive direction or the negative direction of the x-axis.

and let $P(x,y)$ be any point on the locus. Then distance $FP = \sqrt{(x-p)^2 + y^2}$ and distance BP from the y-axis to P is x. Hence, from the definition of a conic section,

$$\left|\frac{FP}{BP}\right| = \left|\frac{\sqrt{(x-p)^2 + y^2}}{x}\right| = e, \tag{2}$$

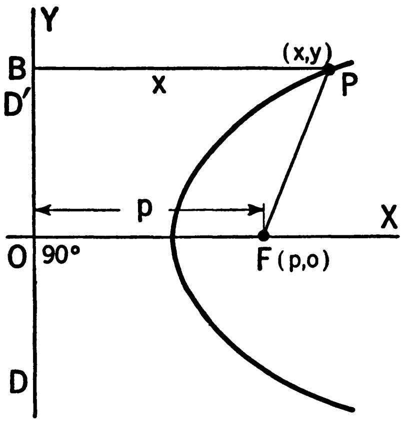


FIG. 6.

where e is a constant. Equating the squares of the two members of (2) and simplifying, we obtain

$$(1-e^2)x^2 + y^2 - 2px + p^2 = 0. \tag{3}$$

We shall find that (3) represents a parabola, an ellipse, or a hyperbola according as $e = 1$, $e < 1$, or $e > 1$.

Example. Discuss the graph of Equation (3) if $e = \sqrt{3}/2$ and $p = 1$.

Solution. Replacing e by $\sqrt{3}/2$ and p by 1 in (3), we obtain

$$(1 - \tfrac{3}{4})x^2 + y^2 - 2x + 1 = 0, \tag{a}$$

or, simplified,

$$x^2 - 8x + 4y^2 = -4. \tag{b}$$

Add $[(\frac{1}{2})8]^2 = 16$ to both members and change slightly to get

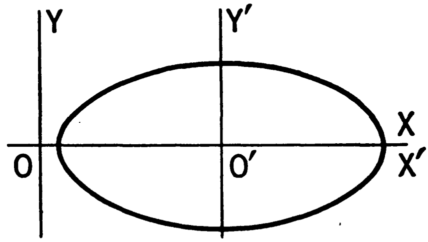


FIG. 7.

$$\frac{(x-4)^2}{12} + \frac{y^2}{3} = 1. \tag{c}$$

Now refer the curve to new axes with origin (4,0) by the substitution

$$x = x' + 4,\ y = y' \tag{d}$$

to obtain

$$\frac{x'^2}{12} + \frac{y'^2}{3} = 1. \tag{e}$$

Observe that the x'- and y'-intercepts are $\pm\sqrt{12}$, $\pm\sqrt{3}$, that the curve is symmetric to both the x'- and y'-axes, and that $|x'| \leqq \sqrt{12}$, $|y'| \leqq \sqrt{3}$. From (e) we get the following table of values:

x'	0	1	2	3	$\sqrt{12}$
y'	$\sqrt{3} = 1.73$	$\frac{1}{2}\sqrt{11} = 1.66$	$\frac{1}{2}\sqrt{8} = 1.41$	$\frac{1}{2}\sqrt{3} = 0.87$	0

Figure 7 shows the graph of (e) and, therefore, of (a).

Exercises

1. In Equation (3) take $e = 1$ and $p = 2$. Then transform the result by the substitution:

$$x = x' + 1, y = y'.$$

Draw both sets of axes and sketch the curve relative to the $x'y'$-axes.

2. Take $e = \sqrt{3}/2$ and $p = \frac{1}{2}$ in (3) and sketch.

3. Take $e = \sqrt{2}$, $p = 2$ in (3). Obtain from the resulting equation by translation of axes a new equation having the form $x'^2 - y'^2 = A$. Draw both sets of axes and sketch the curve.

4. Obtain from Equation (3) by translation of axes a new equation having the form $Ax'^2 + y'^2 = B$.

47. The parabola

The locus of a point which moves so that its distance from a fixed point is equal to its distance from a fixed line not through the point is a parabola. In Figure 8, for a parabola, $FP = BP$ and therefore

$$e = \left|\frac{FP}{BP}\right| = 1. \tag{4}$$

Substituting 1 for e in (3), §46, we obtain as an equation of a parabola

$$y^2 - 2px + p^2 = 0 \tag{5}$$

or

$$y^2 = 2p(x - p/2). \tag{6}$$

The substitution $x = x' + p/2$, $y = y'$ in (6) gives

$$y'^2 = 2px'. \tag{7}$$

Figure 9 shows both sets of axes and the graph of the parabola.

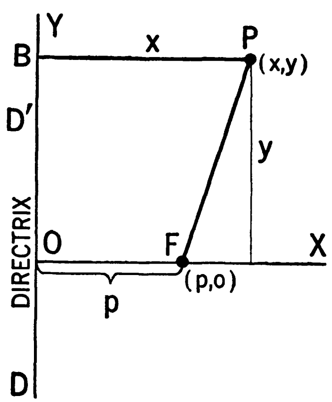


FIG. 8.

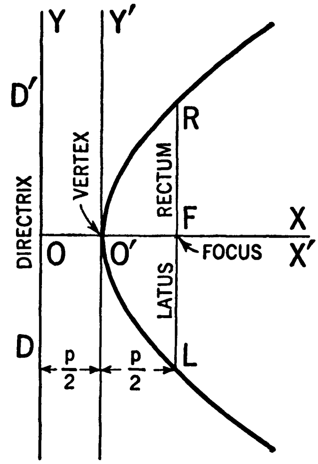


FIG. 9.

The line of symmetry, OX, is called the **axis** of the parabola, and the point of intersection of the parabola and its axis is called the **vertex.** Chords through the focus are called focal chords; the particular one perpendicular to the axis is called the **latus rectum.** Its length is easily found. When $x' = p/2$ in (7), $y' = \pm p$; therefore, the distance FR is $|\,p\,|$ and *the latus rectum has length* $|\,2p\,|$. This fact is useful in sketching a parabola.

If the directrix had been taken as x-axis and the line through the focus perpendicular to the directrix as y-axis, the equation

$$x'^2 = 2py' \tag{8}$$

would have resulted. Forms (7) and (8) are called **standard forms** of the equation of the parabola.

Example 1. Find the coordinates of the focus and the equation of the directrix of the parabola $x^2 = -8y$. Sketch the curve, showing its focus and directrix.

Solution. Since the given parabola is symmetric to the y-axis, we compare with standard form (8) and find that $2p = -8$, $p = -4$, $\frac{1}{2}p = -2$. Therefore, the focus is at the point $\mathbf{(0, -2)}$, and the equation of the directrix is $\boldsymbol{y = 2}$. The latus rectum

extends 4 units either side of the focus and its extremities are the points $(-4,-2)$ and $(4,-2)$. These points, with the vertex, are sufficient for a rough sketch; more points may be easily obtained for a more accurate graph. The graph is shown in Figure 10.

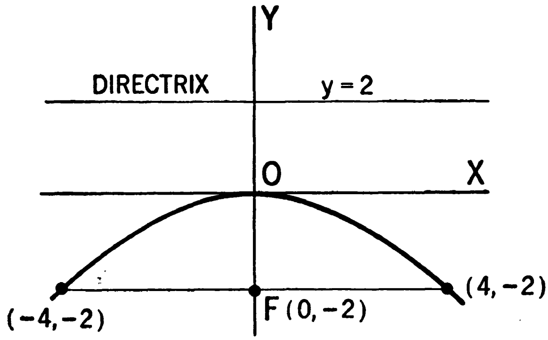


FIG. 10.

Example 2. Find the equation of a parabola with vertex (2,3) and focus (2,1), from the definition of a parabola.

Solution. Figure 11 shows the given vertex and focus, and point $P(x,y)$ representing any point on the required parabola. Since

$$|VF| = \tfrac{1}{2}|p| = 2, \; |p| = 4.$$

Accordingly, the directrix is 4 units from F in the direction of the vertex and its equation is $y = 5$. From the definition of the parabola, distance FP equals the distance from P to directrix, or

$$\sqrt{(x-2)^2 + (y-1)^2} = |5 - y|. \qquad \text{(a)}$$

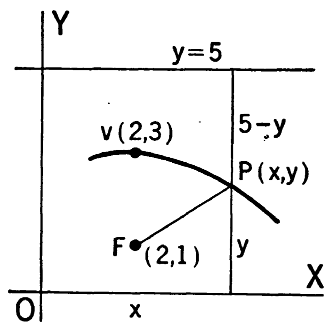


FIG. 11.

From this obtain, by squaring and simplifying,

$$(x-2)^2 + y^2 - 2y + 1 = 25 - 10y + y^2$$

$$(x-2)^2 = -8y + 24 = -8(y-3). \qquad \text{(b)}$$

If we refer the curve to parallel axes with origin at the vertex (2,3), by means of the transformation

$$x = x' + 2, \; y = y' + 3,$$

its equation takes the form

$$x'^2 = -8y'. \qquad \text{(c)}$$

Figure 12 shows the graph and both sets of axes, together with the coordinates of the ends of the latus rectum.

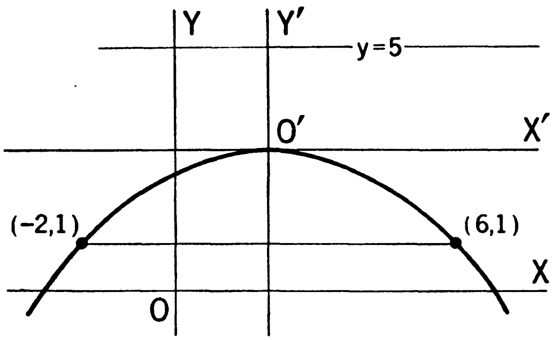


FIG. 12.

Exercises

1. Sketch the graph of $y^2 = 8x$. On your graph mark the focus F; draw the directrix and mark it $D'D''$; draw the latus rectum and mark it $L.\ R.$ Also find the lengths of the double ordinates corresponding to $x = 8$ and $x = 10$.

2. Sketch the graph of $x^2 = 8y$. Mark the focus F, the directrix $D'D''$, and the latus rectum $L.\ R.$ Find the double abscissa corresponding to $y = 50$.

3. Sketch the graph of $y^2 = -4x$.

4. Sketch the graph of $x^2 = -4y$. Find the length of the focal chord of slope $\frac{4}{3}$.

5. Sketch on the same axes the curves $y^2 = 2px$ corresponding to $p = -4, -2, -1, 0, 1, 2, 4$. Describe the effect on the shape of the curve produced by changing p.

6. Find the equation of a parabola with vertex at the origin and passing through the points $(3,2)$ and $(3,-2)$.

7. A parabola has its vertex at the origin and passes through the points $(-6,-6)$ and $(6,-6)$. Find its equation and the coordinates of its focus.

8. Find the equation of a parabola with vertex at the origin and axis along a coordinate axis if the curve passes through $(-3,6)$. How many such parabolas are possible?

9. Prove: Of all the points on a given parabola, the vertex is the one nearest the focus.

10. A circle has its center at the focus of the parabola $y^2 = 4x$ and passes through its vertex. Find the equation of the circle. Where do the circle and parabola intersect?

★11. One end of a focal chord of the parabola $x^2 = 2py$ is the point $(-4,1)$. Find the coordinates of the other end.

12. Two points of the parabola $y^2 = 8x$ have ordinates, respectively, -4 and 8. Find the length of the chord joining them.

13. The main cable of a suspension bridge hangs in the form of a parabola. If the span is 800 ft. and the sag 100 ft., and if the origin is taken at the lowest point of the cable, find an equation of the parabola.

14. Refer the curve $(y - 2)^2 = 8(x - 1)$ to a parallel set of axes with origin at (1,2) and sketch the curve by using the new equation and axes.

15. Write the equation of each curve referred to a parallel set of axes with the indicated origin:

(a) $(y-3)^2=6(x-1)$; (1,3).
(b) $x^2 = 4(y - 2)$; (0,2).
(c) $y^2 + 6y = 2x-1$; $(-4,-3)$.
(d) $(y - k)^2 = 2p(x - h)$; (h,k).
(e) $(x - h)^2 = 2p(y - k)$; (h,k).
(f) $(x - h)^2 = -2p(y - k)$; (h,k).

16. Sketch the parabola $(y - 2)^2 = 4(x - 1)$. Refer it to a parallel set of axes with origin at (1,2). Find the coordinates of its vertex and focus referred to both sets of axes.

17. Sketch the curve $x^2 = 8(y + 2)$. Refer it to a parallel set of axes with origin at $(0,-2)$. Find the coordinates of the focus and of the vertex referred to both sets of axes.

18. Obtain the standard form $y^2 = 2px$, using the definition of a parabola. Take as x-axis the line through the focus perpendicular to the directrix, and as origin the point midway between focus and directrix.

19. Derive the standard form $x^2 = 2py$, using the definition of a parabola. Make a suitable choice of axes.

20. The distance of a point (x,y) from line $x = 6$ is always equal to its distance from point (0,4). Find the equation of the locus of the point.

21. In Exercise 20 replace $x = 6$ by $y = -4$ and (0,4) by (2,0) and solve the resulting problem.

22. Using the definition of a parabola, and the distance formula from §25, find the equation of the locus of a point equidistant from point $(-2,0)$ and line $x + y = 2$.

23. Find the equation of the locus of a point equidistant from (1,0) and $x - y = 6$. Find the vertex.

48. Reduction of the equation of a parabola to standard form

An equation having either of the forms

$$\begin{aligned} ax^2 + bx + cy + d &= 0 \\ ay^2 + bx + cy + d &= 0 \end{aligned} \tag{9}$$

generally represents a parabola. This can be shown by transforming the given equations to one of the standard forms (7) or (8). An example will illustrate the method.

Example. Sketch the parabola $x^2 + 4x + 3y = 8$.

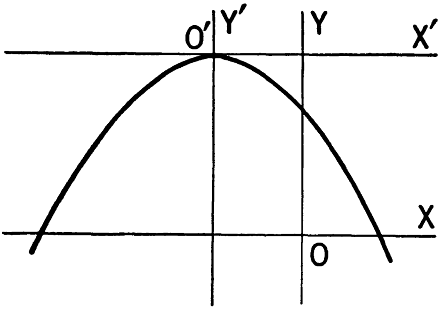


FIG. 13.

Solution. Transfer $3y$ to the right member of the given equation, add 4 to both members of the result, and simplify slightly to obtain

$$x^2 + 4x + 4 = -3y + 8 + 4$$

or

$$(x + 2)^2 = -3(y - 4). \quad \text{(a)}$$

Now let

$$x = x' - 2, \; y = y' + 4 \quad \text{(b)}$$

in (a) to obtain

$$x'^2 = -3y'. \quad \text{(c)}$$

Figure 13 shows the original axes with origin O, the new axes with origin O', and the graph of Equation (c) and therefore of (a).

Exercises

Transform each equation numbered 1 to 4 to the form $x'^2 = \pm ky'$ or $y'^2 = \pm kx'$ and sketch:

1. $y^2 - 2y - 2x = 5$.

2. $x^2 + 2y + 2x - 6 = 0$.

3. $y^2 + 4y + 2x + 2 = 0$.

4. $4x^2 - 12x - 12y + 39 = 0$.

5. Find the equation, referred to the original axes, of the directrix, and the coordinates of focus and of vertex for the parabola of the indicated exercises: (a) 1. (b) 2. (c) 3. (d) 4.

6. Find the equation of a parabola having directrix $x = -2$ and focus (2,3).

7. Find the equation of a parabola having vertex (5,0) and focus (5,3).

8. The distance of a point (x,y) from (5,6) equals its distance from line $y = 2$. Find the equation of its locus and sketch the curve.

9. Figure 14 shows a parabola referred to a polar axis along the axis of the parabola and having the pole at the focus. By expressing the fact that the distances FP and MP are equal, and transforming the result, show that the polar equation of the parabola is:

$$\rho = p/(1 - \cos\theta).$$

Find the polar equation of the parabola having the features numbered 10 to 14.

10. Focus at pole, vertex at (4,180°).

11. Focus at pole, vertex at (4,0°).

12. Focus at pole, vertex at (4,90°).

13. Focus at pole, directrix $\rho \cos \theta = 6$.

14. Focus at pole, directrix $\rho \sin \theta = -6$.

15. Sketch the parabola $\rho = 4/(1 - \cos \theta)$.

16. Sketch the parabola $\rho = 4/(1 + \cos \theta)$.

17. Sketch the parabola $\rho = 4/(1 - \sin \theta)$.

18. Sketch the parabola $\rho = 4/[1 - \cos (\theta + 45°)]$.

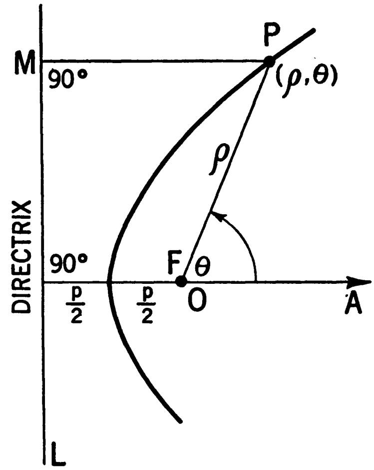


FIG. 14.

19. Transform the polar equation

$$\rho[1 - \cos \theta \cos \alpha - \sin \theta \sin \alpha] = p$$

to rectangular coordinates and transform the result to obtain

$$(x \sin \alpha - y \cos \alpha)^2 = p^2 + 2p(x \cos \alpha + y \sin \alpha).$$

20. Show that $\rho = a \sec^2 (\theta/2)$ represents a parabola.

21. Sketch the parabolas:

(a) $\rho = a \csc^2 (\theta/2)$. (b) $\rho = a \csc^2 [\frac{1}{2}(\theta - \alpha)]$.

★22. Find the equation of the locus of the center of a circle which is always tangent to the line $y = 3$ and to the circle $x^2 + y^2 = 9$. Identify the curve.

★23. Replace $y = 3$ by $x = 1$ in Exercise 22 and solve the resulting problem.

★24. Replace $y = 3$ by $x = -5$ in Exercise 22 and solve the resulting problem.

★25. Find the equation of the locus of a point which is always four units farther from (2,0) than from line $x = 0$.

Hint. If F represents the given point and $P(x,y)$ any point on the locus for which $x > 0$, $|FP| = x + 4$; if $x < 0$, $|FP| = -x + 4$.

Figure 15 represents the locus.

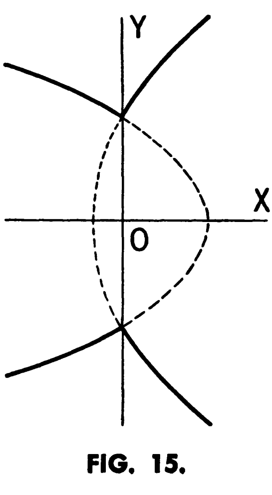


FIG. 15.

★26. Show that the length of any focal chord of a parabola is twice the distance of its midpoint from the directrix.

★27. Find the locus of the midpoints of the focal chords of the parabola $\rho = p/(1 - \cos \theta.)$

49. Parabolas in nature and in applications

If an object were thrown from the ground in a vacuum, it would describe a curve closely approximating the parabola. In fact, parabolas are first approximations to the curves described by objects thrown into the air, and by the projectiles of guns.

FIG. 16.

Many comets describe curves that are roughly parabolas with the sun as their focus.

The parabolic form is frequently used in applications. If an arch (see Figure 16) is to support a uniform horizontal load,

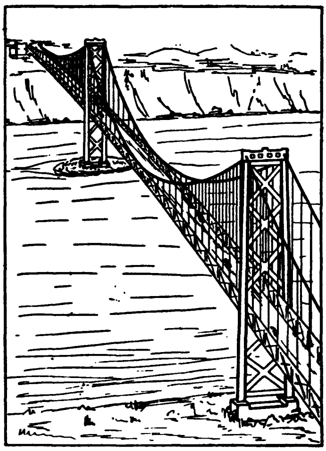


FIG. 17.

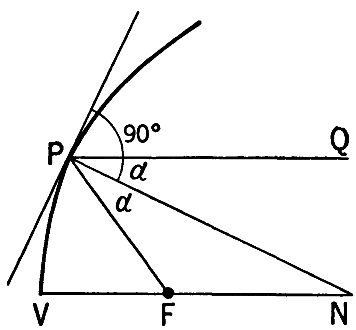


FIG. 18.

there is a certain parabolic shape for the arch such that the stress will be directed tangent to the curve of the arch. Similarly, the main cables of a suspension bridge (see Figure 17) designed to carry uniform horizontal loads hang in curves roughly parabolic in shape.

A property of the parabola illustrated in Figure 18 is basic in the design of reflectors of sound and light. If P represents any point on the parabola (see Figure 18), F represents its focus, and PQ is parallel to the axis FN, then the normal PN perpendicular to the tangent at P bisects the angle FPQ. A ray of light from F to P will be reflected along the line PQ. If now a reflector has the shape obtained by revolving a parabola about its axis, the rays from a light at the focus will be reflected parallel to the axis, thus producing a concentrated beam instead of a diffused light. The same principle applies to devices for directing and concentrating sound waves.

50. The ellipse

The locus of a point which moves so that its distance from a fixed point divided by its distance from a fixed line not through the point is a constant less than unity is an **ellipse.**

Equation (3), §**46,** namely,

$$(1 - e^2)x^2 + y^2 - 2xp + p^2 = 0, \tag{10}$$

represents an ellipse if $e < 1$. Changing the axes of reference in (10) by the substitution

$$x = x' + \frac{p}{1 - e^2}, \; y = y', \tag{11}$$

obtain

$$(1 - e^2)\left[x'^2 + \frac{2p}{1 - e^2}x' + \frac{p^2}{(1 - e^2)^2}\right] - 2p\left(x' + \frac{p}{1 - e^2}\right) + y'^2 + p^2 = 0,$$

or, simplifying,

$$(1 - e^2)x'^2 + y'^2 = -p^2 + \frac{2p^2}{1 - e^2} - \frac{p^2}{1 - e^2} = \frac{p^2e^2}{1 - \rho^2}. \tag{12}$$

Dividing (12) through by $p^2e^2/(1 - e^2)$, obtain

$$\frac{x'^2}{\dfrac{p^2e^2}{(1 - e^2)^2}} + \frac{y'^2}{\dfrac{p^2e^2}{1 - e^2}} = 1. \tag{13}$$

In (13) let

$$a = \frac{pe}{1 - e^2}, \; b = \frac{pe}{\sqrt{1 - e^2}}, \tag{14}$$

to obtain as the transformed equation of the ellipse (10)

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1. \tag{15}$$

The graph of (15) has intercepts $x' = \pm a$, $y' = \pm b$, it is symmetric with respect to both the x'-axis and the y'-axis, $|\,x'\,| \leqq a$, $|\,y'\,| \leqq b$, and the following table of values applies:

x'	0	$a/4$	$2a/4$	$3a/4$	a
y'	b	$\frac{b\sqrt{15}}{4} = .97b$	$\frac{b\sqrt{3}}{2} = .87b$	$\frac{b\sqrt{7}}{4} = .66b$	0

Figure 19 represents the graph of (15), both sets of axes, and other essential features. The distance $OC = p/(1 - e^2)$ was

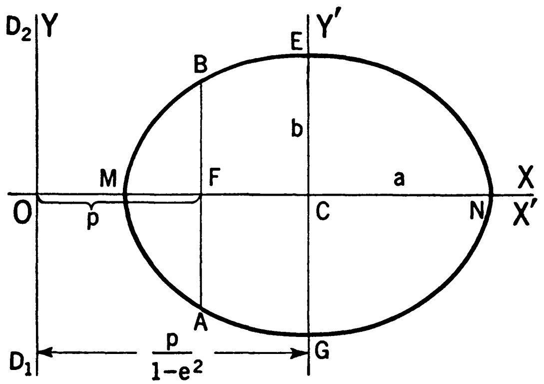


FIG. 19.

obtained from (11), and this, from the first equation of (14), is a/e. The chord MN, through F perpendicular to D_1D_2, is called the **major axis** of the ellipse; C, the midpoint of MN, is called the **center**; and the chord GE, through C perpendicular to MN, is called the **minor axis** of the ellipse. The points M and N common to the major axis and the ellipse are called the **vertices.** The chord AB in Figure 19, through the focus perpendicular to the major axis, is called the **latus rectum.**

Figure 20 shows the graph, and a number of features which will now be considered. From Figure 19 and the expression for a in Equation (14),

$$FC = \frac{p}{1 - e^2} - p = \frac{pe^2}{1 - e^2} = ae. \tag{16}$$

From Figure 20 and Equation (16),

$$\overline{FE}^2 = a^2e^2 + b^2 = \frac{p^2e^4}{(1 - e^2)^2} + \frac{p^2e^2}{1 - e^2} \cdot \frac{1 - e^2}{1 - e^2} = \frac{p^2e^2}{(1 - e^2)^2} = a^2,$$

and

$$FE = a. \tag{17}$$

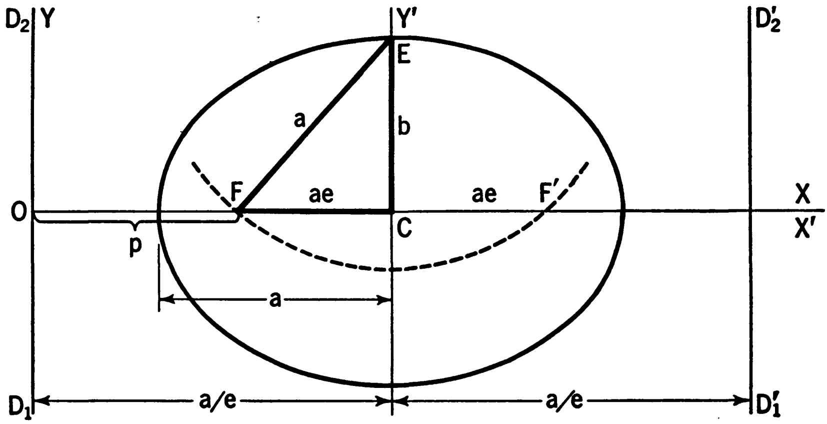


FIG. 20.

The relations shown in Figure 20, especially the relations involved in the triangle FEC, are interesting and fairly important.

The symmetry of the graph, deduced from Equation (15), enables us to see that there is a focus F' and directrix $D_1'D_2'$ symmetric, respectively, to the corresponding F and D_1D_2 with respect to the minor axis. Example 2 below states another important relation.

From Equations (14) it appears that

$$a = \frac{b}{\sqrt{1 - e^2}} > b, \tag{18}$$

and, therefore, that the major axis containing the foci must be greater than the minor axis. Hence, *the graph of an equation*

$$\frac{\mathbf{x}^2}{\mathbf{m}^2} + \frac{\mathbf{y}^2}{\mathbf{n}^2} = \mathbf{1, n > m} \tag{19}$$

has its foci on the y-*axis,* and the graph is represented by a figure such as 21. Equation (15) without the primes and Equation (19) *will be called standard forms of the equations of ellipses.*

Example 1. For the ellipse $9x^2 + 25y^2 = 225$, draw the graph and the two directrices, and locate the foci.

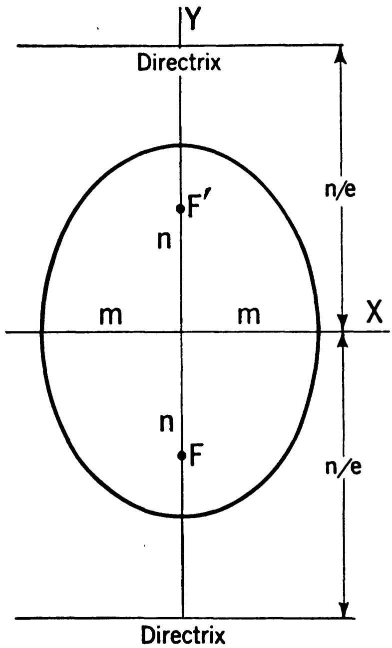


FIG. 21.

Solution. Dividing the given equation through by 225, we obtain

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

Plotting in the usual way, we obtain the ellipse shown in Figure 22. Evidently, $a = 5$, $b = 3$. Hence, we construct the triangle FCE with $CE = b = 3$ and $FE = a = 5$. Then, $FC = ae = 5e = 4$, and $e = \frac{4}{5}$.

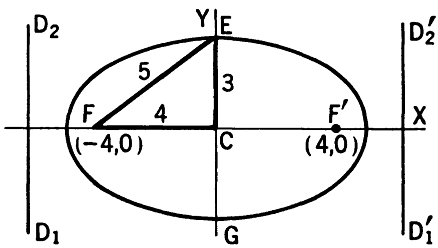


FIG. 22.

The distance from center to directrix is $a/e = 5/(\frac{4}{5}) = \frac{25}{4}$. Figure 22 shows the required lines, points, and distances.

Example 2. If $P(x,y)$ is any point on ellipse $x^2/a^2 + y^2/b^2 = 1$, with a the semi-major axis, and if F and F' are the foci (see Figure 23), show that one of the lines FP and $F'P$ has a length $a + ex$, the other the length $a - ex$, and, therefore, that

$$FP + F'P = 2a.$$

Solution. From Figure 23, the equation of the ellipse, and the relation $b^2 = a^2 - a^2e^2$,

$$\begin{aligned} FP &= \sqrt{(x + ae)^2 + y^2} = \sqrt{(x + ae)^2 + b^2(1 - x^2/a^2)} \\ &= \sqrt{x^2 + 2aex + a^2e^2 + (a^2 - a^2e^2)(1 - x^2/a^2)} = a + ex. \quad \text{(a)} \end{aligned}$$

Similarly,

$$F'P = \sqrt{(x - ae)^2 + y^2} = \sqrt{(x - ae)^2 + (a^2 - a^2e^2)(1 - x^2/a^2)}$$
$$= a - ex. \quad \text{(b)}$$

From (a) and (b),

$$FP + F'P = a + ex + a - ex = \mathbf{2a}.$$

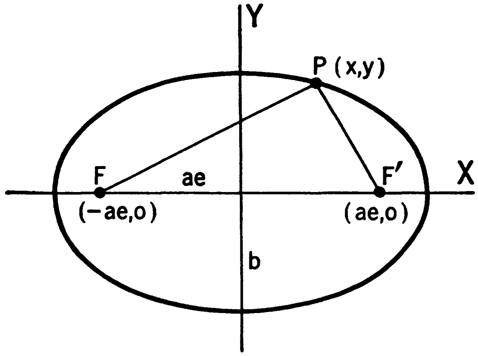


FIG. 23.

A definition of an ellipse based on this property follows: *An ellipse is the locus of a point which moves so that the sum of its distances from two fixed points is a constant* 2a.

Exercises

For each of the equations numbered 1 to 4, sketch the ellipse represented, draw its directrices, indicate its foci, and find the length of its latus rectum:

1. $16x^2 + 25y^2 = 400.$ **3.** $x^2 + 4y^2 = 4.$

2. $25x^2 + 16y^2 = 400.$ **4.** $4x^2 + y^2 = 4.$

Write the coordinates of the foci and the equations of the directrices for each ellipse represented by the equations numbered 5 to 8:

5. $4x^2 + y^2 = 64.$ **7.** $9x^2 + 4y^2 = 36.$

6. $x^2 + 4y^2 = 36.$ **8.** $7x^2 + 8y^2 = 112.$

Transform each of the equations numbered 9 to 12 to standard form using translation of axes, draw both sets of axes, and sketch the graph:

9. $9(x - 1)^2 + (y - 2)^2 = 9.$ **11.** $25x^2 + (y - 5)^2 = 100.$

10. $(x + 1)^2 + 4y^2 = 4.$ **12.** $4(x + 1)^2 + (y + 2)^2 = 9.$

13. If e is nearly zero, would you expect the corresponding ellipse to be long and thin, or nearly round? If e is nearly 1, what would you expect?

14. What is the eccentricity of a circle considered as an ellipse? Does it have directrices?

15. The orbit of the earth about the sun is an ellipse having an eccentricity of $\frac{1}{60}$ approximately. Is the orbit nearly round or long and thin?

Find the equations of the ellipses having the origin as center and otherwise defined by the conditions numbered 16 to 20:

16. Vertices $(\pm 5,0)$, passing through $(4,\frac{12}{5})$.

17. One vertex at $(6,0)$, minor axis 6.

18. One vertex at $(10,0)$, $e = \frac{3}{5}$.

19. One focus at $(4,0)$, $e = \frac{1}{2}$.

20. Having foci on the x-axis, and passing through $(2,1)$ and $(\sqrt{2},\sqrt{2})$.

21. A rectangle inscribed in the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ has a pair of opposite sides passing through the foci. Find its dimensions.

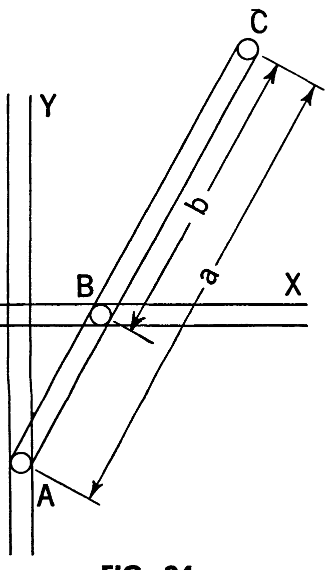


FIG. 24.

22. Find the length of the diagonal of a square inscribed in the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

23. Find the length of the latus rectum of the ellipse $x^2 + 4y^2 = 25$ and the length of the chord cut from the ellipse by the line $y = 2$.

24. A chord of ellipse $4x^2 + 5y^2 = 20$ passes through a focus and one end of the minor axis. Find its length.

25. Find the locus of a point for which the sum of its distances from: (a) $(4,0)$ and $(-4,0)$ is 10; (b) $(0,3)$ and $(0,-3)$ is 10.

26. It is shown in calculus that the slope of the tangent to ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at point (X,Y) is $-b^2X/(a^2Y)$. Assuming this, find the equation of the tangent to $x^2 + 4y^2 = 8$ at $(2,1)$.

27. A point P is situated on the line segment AB, of length a, one-third of the way from A to B. Find the locus of point P if the end A of the segment must always be on the y-axis and end B on the x-axis.

28. Show that if A, B, and C are three points fixed on a rod (see Figure 24), if $BC = b$ and $AC = a$, if A must move on the y-axis and B on the x-axis, then the locus of C is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The ellipsograph, an instrument for drawing ellipses, applies this principle.

29. In the circle $x^2 + y^2 = a^2$, every chord perpendicular to the y-axis is divided into three equal parts. Find the equation of the locus of the points of division.

30. Find the locus of the midpoints of ordinates, for each of the curves: (a) $x^2 + y^2 = 100$; (b) $b^2x^2 + a^2y^2 = a^2b^2$.

31. Find the locus of a point P if the product of the slopes of the lines connecting P to $(-a,0)$ and $(a,0)$ is $-b^2/a^2$.

32. A circle with center at a focus of an ellipse having $e > \frac{1}{2}$ passes through the center of the ellipse and meets it in points A and B. Show that the length of a line segment from A or B to the directrix corresponding to the focus mentioned is that of the semi-major axis.

51. Ellipse referred to axes parallel to its axes

Consider the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1. \tag{20}$$

Apply the transformation

$$x = x' + h,\ y = y' + k \tag{21}$$

to Equation (20) to obtain

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1, \tag{22}$$

a standard form of the equation of an ellipse. This procedure shows that *Equation* (20) *is the equation of an ellipse having center* (h,k) *and having semi-axes of lengths* a *and* b *parallel to the coordinate axes* (see Figure 25).

In general, the equation

$$\text{A}x^2 + \text{B}y^2 + \text{C}x + \text{D}y + \text{E} = 0 \tag{23}$$

represents an ellipse if A *and* B *have the same sign.* This can be shown by transforming the given equation to standard form (20). The following example will illustrate the procedure.

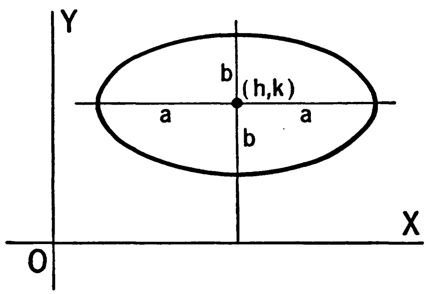


FIG. 25.

Example. Sketch the ellipse

$$9x^2 + 4y^2 - 18x + 16y = 11. \tag{a}$$

Solution. Write Equation (a) in the form

$$9(x^2 - 2x) + 4(y^2 + 4y) = 11, \qquad \text{(b)}$$

add 1 inside the first parenthesis, 4 inside the second parenthesis, and then $9 \times 1 + 4 \times 4 = 25$ to the right member to obtain

$$9(x - 1)^2 + 4(y + 2)^2 = 36, \qquad \text{(c)}$$

or, dividing through by 36,

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1. \qquad \text{(d)}$$

The graph may be drawn directly from (d), or we may transform it, by using $x = x' + 1$, $y = y' - 2$, to the form

$$\frac{x'^2}{4} + \frac{y'^2}{9} = 1, \qquad \text{(e)}$$

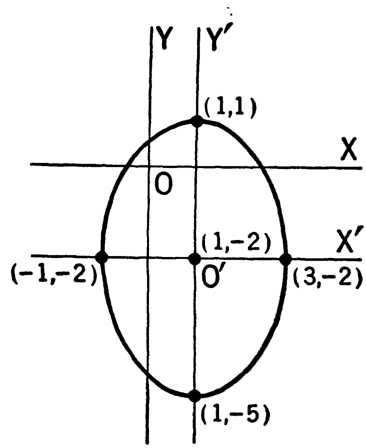


FIG. 26.

draw both sets of axes, and then plot the curve from (e). Figure 26 shows the required graph.

Exercises

Each set of data numbered 1 to 4 applies to two ellipses having axes parallel to the coordinate axes. Find the equations of the two ellipses in each case:

1. Center (1,3), major axis 10, minor axis 6.

2. Center (0,4), major axis 8, minor axis 2.

3. Center (5,0), major axis 10, minor axis 6.

4. Center $(h,0)$, major axis $2a$, minor axis $2b$.

5. Find the coordinates of the foci of the ellipse (a) of the example when referred to the $x'y'$-axes, and also when referred to the original axes.

Sketch each of the curves numbered 6 to 9. In each case give the coordinates of its foci referred to the original coordinate axes.

6. $x^2 + 4y^2 - 2x - 8y + 1 = 0.$

7. $4x^2 + y^2 + 6y + 5 = 0.$

8. $25x^2 + 9y^2 - 150x = 0.$

9. $x^2 + 4y^2 + 12y + 5 = 0.$

For each set of data numbered 10 to 15, find an equation of the corresponding ellipse.

10. Center (2,2), foci $(2 \pm 3,2)$, minor axis 8.

11. Center $(a,0)$, foci on the x-axis, semi-axes 5 and 3.

12. Foci $(\pm 3,0)$, length of major axis 10.

13. Foci $(0, \pm 4)$, length of major axis 10.

14. Vertices $(\pm 5,0)$, length of minor axis 8.

15. Vertices $(0, \pm 5)$, directrices $y = \pm \frac{25}{3}$.

16. Find the equation of all ellipses having length of minor axis 10, one focus at the origin, and the other focus on: (a) the y-axis; (b) the x-axis.

17. In the ellipse $(x - 2)^2 + 4(y + 2)^2 = 4$, every chord perpendicular to the x-axis is divided into three equal parts. Find the equation of the locus of the points of division.

18. Find the equation of the locus of a point having the sum of its distances from $(-3,-3)$ and $(5,-3)$ equal to 10.

19. Find the locus of a point P if the product of the slopes of the lines connecting P to points $(h - c,k)$ and $(h + c,k)$ is $-b^2/a^2$.

★20. Show that if (x_1,y_1), (x_2,y_2), and (x_3,y_3) are three points not on the same straight line, and a and b are any two numbers having the same sign, then

$$\begin{vmatrix} ax^2 + by^2 & x & y & 1 \\ ax_1^2 + by_1^2 & x_1 & y_1 & 1 \\ ax_2^2 + by_2^2 & x_2 & y_2 & 1 \\ ax_3^2 + by_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

represents an ellipse through the given points.

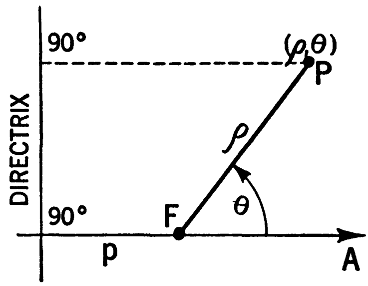


FIG. 27.

21. Taking the focus as pole and a line perpendicular to the directrix as polar axis (see Figure 27), show that the polar equation of the ellipse is

$$\rho/(p + \rho \cos \theta) = e, \text{ or}$$

$$\rho = \frac{pe}{1 - e \cos \theta}.$$

22. Comparing each equation with the last one in Exercise 21, find e and p for the ellipse represented:

(a) $\rho(1 - \frac{1}{2} \cos \theta) = 5$.

(b) $\rho(3 - 2 \cos \theta) = 9$.

(c) $\rho(m - n \cos \theta) = g$, $|n| < |m|$.

For the ellipses represented in the exercises numbered 23 and 24, find the intercepts on the polar axis and on $\theta = 90°$ by letting $\theta = 0$, $\theta = 180°$, $\theta = 90°$, and $\theta = 270°$ in order. Plot the corresponding points, sketch the ellipse, and find the length of its axes.

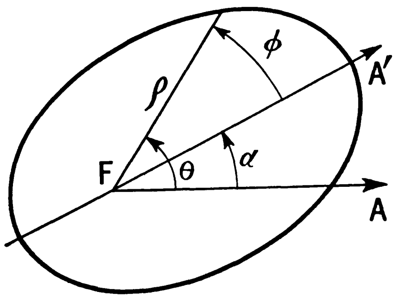


FIG. 28.

23. $\rho(3 - \cos\theta) = 12.$

24. $\rho(5 - 2\cos\theta) = 105.$

25. Using the method of §**37** and the result of Exercise 21, show that the equation of an ellipse with a focus F at the pole and with its major axis along a directed line FA' which makes an angle α with the polar axis FA (see Figure 28) is

$$\rho = \frac{pe}{1 - e\cos(\theta - \alpha)}.$$

26. Using the equation in Exercise 25 with $\alpha = 0°$, $180°$, $90°$, and $270°$ in turn, derive the following equations: $\rho(1 \mp e\cos\theta) = pe$, $\rho(1 \mp e\sin\theta) = pe$, and sketch the curves.

For the graphs represented in the exercises 27 to 30, find the intercepts on the lines $\theta = 0°$ and $\theta = 90°$ and sketch the graphs. Find the lengths of the axes of each ellipse:

27. $\rho = \dfrac{12}{2 + \sin\theta}.$

28. $\rho = \dfrac{28}{4 - 3\sin\theta}.$

29. $\rho = \dfrac{20}{5 + 3\cos\theta}.$

30. $\rho = \dfrac{20}{5 + 3\cos(\theta - 30°)}.$

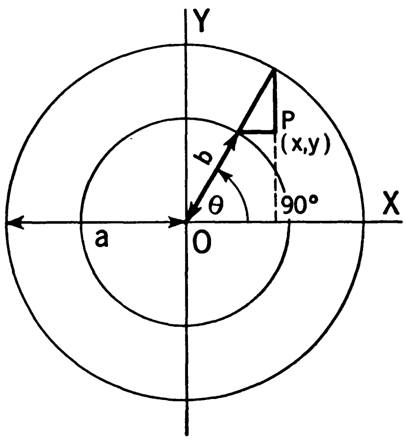


FIG. 29.

31. Figure 29 represents two concentric circles of radii a and b. From the figure show that the coordinates of $P(x,y)$ are:

$x = a\cos\phi,\qquad y = b\sin\phi.$

Eliminate ϕ from these equations and deduce from the result that the locus of P is an ellipse. Sketch both circles and the ellipse.

32. Find in rectangular form the equation of the ellipse defined by $x = h + a\cos\phi$, $y = k + b\sin\phi$. Describe the ellipse.

52. Nature's ellipses. Applications

The ellipse is a curve highly favored by nature. The natural path of a planet moving under the gravitational pull of a sun

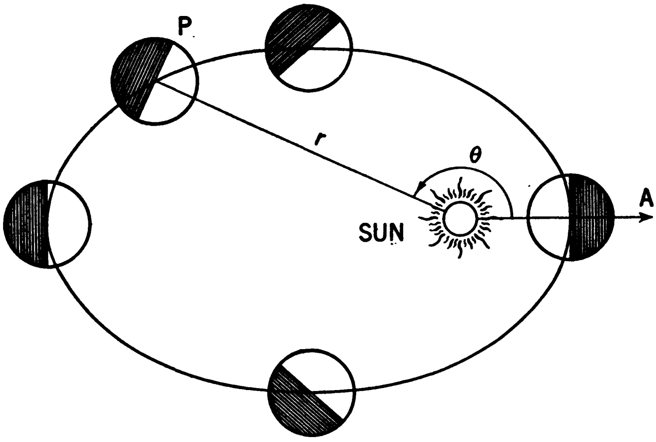


FIG. 30.

(see Figure 30) is an ellipse. Thus, the earth moves relative to the sun approximately in an ellipse with the sun at one focus; the moon moves relative to the earth in an ellipse with the earth at its focus.

There are numerous applications of ellipses in science, engineering, and architectural design. They are used, for example, in gears of varying angular velocity (see Figure 31) and in semi-elliptic springs. The ellipse is frequently used to represent the distribution of stress in bodies. A very interesting property of an ellipse is that the normal at any point on it bisects the angle between the lines connecting the point to the foci of the ellipse. (See Figure 32.) Consequently, a surface formed by revolving an ellipse about its major axis has the property that a light wave or sound wave from one focus will be reflected by the surface so as to arrive at the other focus. The whispering

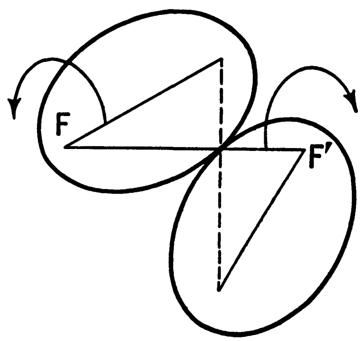


FIG. 31. For elliptical gears, the equal ellipses are pivoted at foci F and F', and FF' is equal to the major axis.

galleries like the one in the dome of the Capitol Building at Washington have ceilings in the form of an ellipsoid of revolution. In the Cupola of St. Paul's, London, a low whisper near one wall can be distinctly heard at the opposite wall 108 feet away.

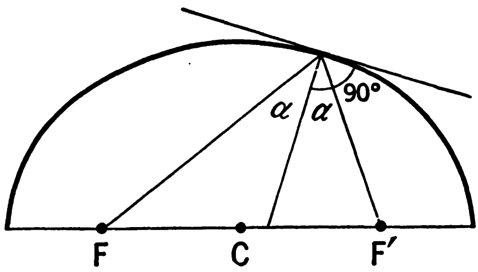


FIG. 32.

53. The hyperbola

The locus of a point which moves so that its distance from a fixed point divided by its distance from a fixed line not through the point is a constant greater than unity, is a **hyperbola.**

Equation (3), §46, multiplied through by -1 is

$$(e^2 - 1)x^2 - y^2 + 2px - p^2 = 0. \tag{24}$$

It applies for the hyperbola when $e > 1$. Changing the axes of reference in (24) by the substitution

$$x = x' - \frac{p}{e^2 - 1}, \; y = y', \tag{25}$$

obtain from (24), after some simplification,

$$\frac{x'^2}{p^2e^2/(e^2 - 1)^2} - \frac{y'^2}{p^2e^2/(e^2 - 1)} = 1. \tag{26}$$

In (26) let

$$a = \frac{pe}{e^2 - 1}, \; b = \frac{pe}{\sqrt{e^2 - 1}} \tag{27}$$

to obtain as the transformed equation of the hyperbola (24)

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1. \tag{28}$$

This equation and the one obtained from it by interchanging x' and y' will be called the *standard forms of the hyperbola.* The curve is symmetric with respect to both the x'-axis and the y'-axis. The x'-intercepts are $\pm a$, and the y'-intercepts are imaginary. Solve (28) for y' to obtain:

$$\frac{y'^2}{b^2} = \frac{x'^2}{a^2} - 1, \text{ or } y' = \pm\frac{b}{a}\sqrt{x'^2 - a^2}. \tag{29}$$

From this we see that $x'^2 - a^2$ must not be negative. Hence,

$$x' \geqq a \text{ or } x' \leqq -a.$$

Using these facts and the following table of values obtained from (29),

x'	$\pm a$	$\pm 2a$	$\pm 3a$	$\pm 4a$
y'	0	$\pm\sqrt{3}\,b$	$\pm\sqrt{8}\,b$	$\pm\sqrt{15}\,b$

we plot the curve shown in Figure 33. From the first equation of (29), it appears that $|y'/b| < |x'/a|$, but the two values approach equality as x', and therefore y', increases. In other words, when x' is large, the 1 in the first equation of (29) is insignificant in comparison with the values of x'^2/a^2 and y'^2/b^2. Omitting the 1, we have, as an approximation to the curve for large values of x',

$$\frac{y'^2}{b^2} = \frac{x'^2}{a^2}, \text{ or}$$

$$y' = \pm \frac{b}{a} x'. \qquad (30)$$

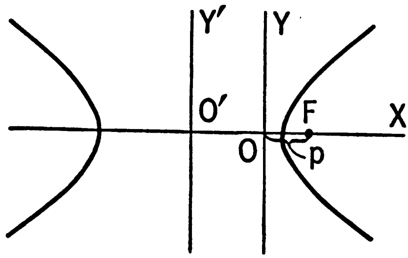


FIG. 33.

These lines are called the *asymptotes* of the hyperbola. Figure 34 shows the hyperbola (29) with its asymptotes. A good sketch of the hyperbola is obtained by drawing the rectangle with dimensions $2a$ and $2b$, then its diagonals, and finally the hyperbola.

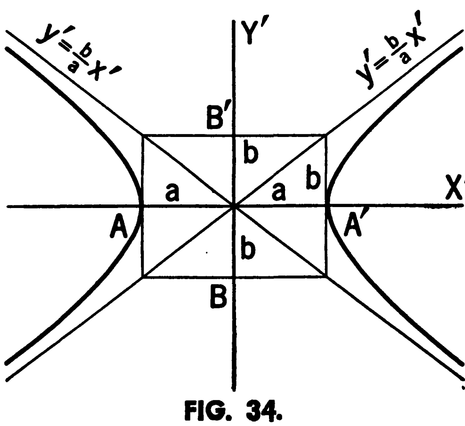


FIG. 34.

The points A and A' in Figure 34, where an axis of symmetry cuts the hyperbola, are called the **vertices,** the part AA' of this axis between the vertices is called the **transverse axis,** and the part BB' is called the **conjugate axis.** The lengths $2a$ and $2b$ are called the lengths of the transverse axis and of the conjugate axis.

From (25) we deduce that the distance $O'O$ in Figure 35, from center to directrix, is $p/(e^2 - 1)$, and this, from (27), is a/e. From Figure 35,

$$O'F = O'O + OF,$$

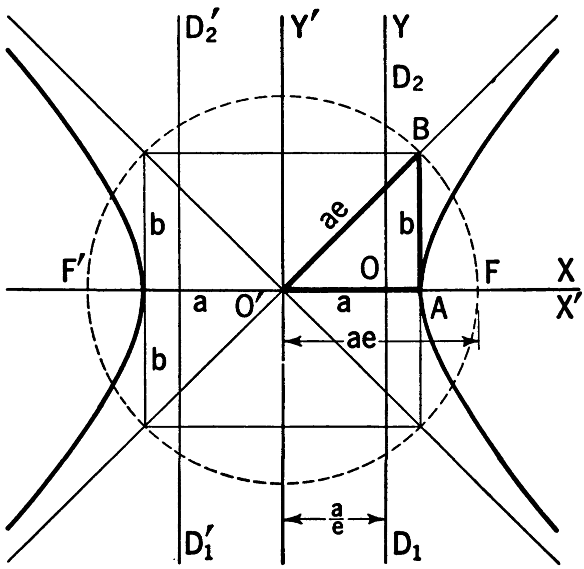


FIG. 35.

and therefore,

$$O'F = \frac{p}{e^2 - 1} + p = \frac{pe^2}{e^2 - 1} = ae.$$

From (27),

$$(O'B)^2 = a^2 + b^2 = \frac{p^2e^2}{(e^2 - 1)^2} + \frac{p^2e^2}{e^2 - 1} \cdot \frac{e^2 - 1}{e^2 - 1} = \frac{p^2e^4}{(e^2 - 1)^2}$$
$$= a^2e^2.$$

The last two equations show that $O'B = O'F = ae$.

Note the triangle $O'AB$, from which the relation $a^2e^2 = a^2 + b^2$ can be read.

The symmetry of the graph of Equation (28) enables us to see that there is a focus F' and a directrix $D_1'D_2'$ symmetric, respectively, to the corresponding F and D_1D_2 with respect to the conjugate axis (see Figure 35).

A hyperbola may have its transverse axis along the y-axis. Example 1 which follows will consider such a hyperbola. The two hyperbolas represented by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{31}$$

are called **conjugate hyperbolas.** Figure 36 shows the first in full line and the second in dotted line. They have the same asymp-

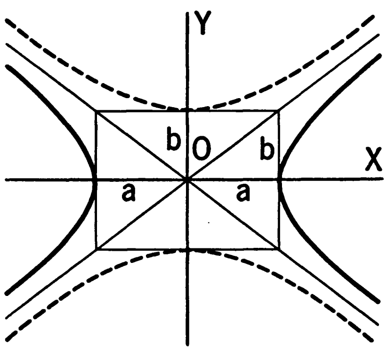


FIG. 36.

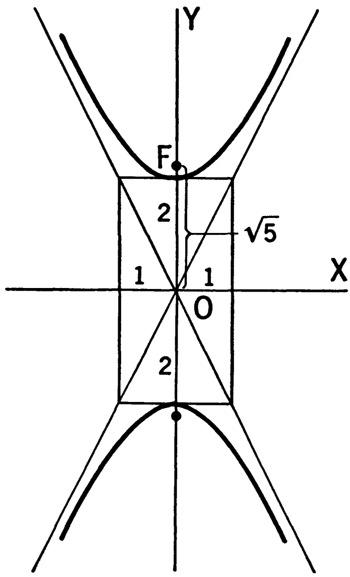


FIG. 37.

totes, but their eccentricities are different and the transverse axis of either is the conjugate axis of the other.

Example 1. Sketch the curve $y^2 - 4x^2 = 4$, showing foci and asymptotes. Write the equations of the directrices.

Solution. Division of the given equation by 4 gives

$$\frac{y^2}{4} - \frac{x^2}{1} = 1. \tag{a}$$

The y-intercepts, ± 2, are real and the vertices are on the y-axis.

The x-intercepts are imaginary and $b = 1$. The equations of the asymptotes, got by replacing 1 by zero in (a), are $y = \pm 2x$. From the given equation, write $x = \pm\frac{1}{2}\sqrt{y^2 - 4}$ and compute the table:

x	0	$\pm\sqrt{5}/2$	$\pm\sqrt{12}/2$	$\pm\sqrt{21}/2$
y	± 2	± 3	± 4	± 5

Figure 37 shows the required graph. Since $a^2 + b^2 = a^2e^2$, $5 = 4e^2$ and $e = \sqrt{5}/2$. Hence, the distance from the center to

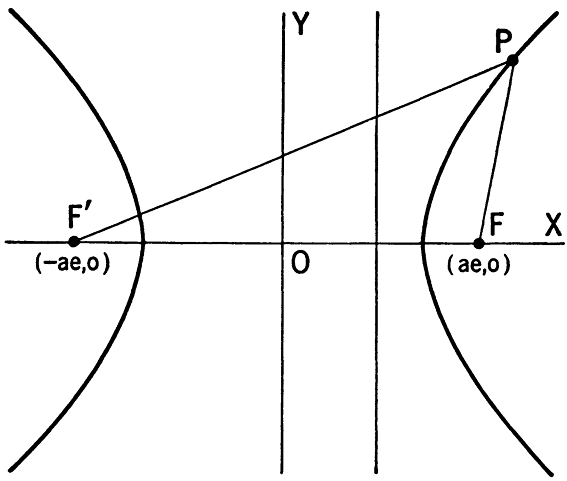


FIG. 38.

a directrix is $a/e = 2/(\sqrt{5}/2) = 4/\sqrt{5}$, and the equations of the directrices are $y = \pm 4/\sqrt{5}$.

Example 2. If $P(x,y)$ is any point on the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$, and $F(ae,0)$ and $F'(-ae,0)$ are its foci (see Figure 38), show that the length of one of the lines FP and $F'P$ is $|ex + a|$ and the length of the other is $|ex - a|$, and that the difference of the lengths of FP and $F'P$ is $2a$.

Solution. From Figure 38, the equation of the hyperbola, and the relation $a^2 + b^2 = a^2e^2$, we obtain

$$\begin{aligned} FP &= \sqrt{(x - ae)^2 + y^2} = \sqrt{(x - ae)^2 + b^2\,[(x^2/a^2) - 1]} \\ &= \sqrt{x^2 - 2aex + a^2e^2 + (a^2e^2 - a^2)(x^2 - a^2)/a^2} \\ &= ex - a \end{aligned} \tag{a}$$

when x is positive. Similarly,

$$F'P = \sqrt{(x + ae)^2 + (a^2e^2 - a^2)(x^2 - a^2)/a^2} = ex + a \quad \text{(b)}$$

when x is positive. Hence, on the right half of the hyperbola,

$$F'P - FP = (ex + a) - (ex - a) = 2a.$$

Inspection shows that when P is on the left half of the hyperbola, and x is consequently negative, $F'P$ and FP are $-(ex + a)$ and $-(ex - a)$, respectively, and $F'P - FP = -2a$. Hence, whether x is positive or negative,

$$|F'P - FP| = 2a.$$

A definition of a hyperbola based on this properly follows. *A hyperbola is the locus of a point which moves so that the difference of its distances from two fixed points is a constant* 2a.

Exercises

For each of the equations numbered 1 to 4, sketch the hyperbola represented, draw its asymptotes and directrices, and find the length of the latus rectum (the chord through a focus perpendicular to the transverse axis).

1. $9x^2 - 16y^2 = 144.$

2. $9x^2 - y^2 = 9.$

3. $16y^2 - 9x^2 = 144.$

4. $y^2 - x^2 = 9.$

For each set of data numbered 5 to 10, find an equation of the corresponding hyperbola:

5. Foci, $(\pm 5,0)$; length of transverse axis, 8.

6. Foci on the x-axis, lengths of transverse axis and conjugate axis 6 and 8, respectively, and center at the origin.

7. Foci on the y-axis, lengths of transverse axis and conjugate axis 8 and 6, respectively, and center at the origin.

8. Foci, $(0, \pm 5)$; length of transverse axis, 8.

9. Foci, $(\pm 5,0)$; directrices, $x = \pm 4$.

10. Vertices, $(0, \pm 4)$; directrices, $y = \pm 3$.

11. Consider the triangle $O'AB$ of Figure 35 and determine whether a hyperbola bends sharply or slowly at the vertex when: (a) e is nearly unity; (b) e is very large.

12. If a and b in the first equation of (31) vary and approach zero indefinitely close while e remains constant, what does the hyperbola approach? Answer the same question for the second hyperbola of (31).

Sketch the graph of each of the hyperbolas numbered 13 to 16 and its conjugate. Also give eccentricity, length of transverse axis, and distance from center to directrix of each hyperbola and its conjugate:

13. $x^2 - 4y^2 = 4.$

14. $4y^2 - 9x^2 = 36.$

15. $x^2 - y^2 = 4.$

16. $3y^2 - 2x^2 = 10.$

17. Find the locus of midpoints of ordinates of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$.

18. Find the length of a chord through a focus of $b^2y^2 - a^2x^2 = a^2b^2$ and perpendicular to the transverse axis.

19. Find the length of the chord of $4x^2 - y^2 = 12$ through points $(2,-2)$ and $(\frac{5}{2},0)$.

20. Show that the distance from a focus to an asymptote of $b^2x^2 - a^2y^2 = a^2b^2$ is equal to b.

21. Find the length of the edge of a square inscribed in $4x^2 - y^2 = 4$.

22. If P is any point on a hyperbola, prove that the product d_1d_2 of its distances d_1 and d_2 from the asymptotes of the hyperbola is constant.

23. Find the locus of a point if the difference of its distances from $(5,0)$ and $(-5,0)$ is 8.

24. Find the locus of a point if the difference of its distances from $(0,c)$ and $(0,-c)$ is $2a$.

25. It is shown in calculus that the slope of a tangent to $b^2x^2 - a^2y^2 = a^2b^2$ at a point (X,Y) on it is b^2X/a^2Y. Find the equation of the tangent to $x^2 - 4y^2 = 12$ at $(4,1)$.

26. If the chords of $y^2 - 4x^2 = 12$ cut out by lines perpendicular to the y-axis are divided into three equal parts, find the locus of the points of division.

27. Find the locus of point P if the product of the slopes of the lines connecting P to the points $(0,3)$ and $(0,-3)$ is equal to 2.

28. A circle with center at a focus of a hyperbola passes through the center of the hyperbola and meets it in points A and B. Show that the length of a line segment from A or B to the directrix, corresponding to the focus mentioned, is that of the semi-transverse axis.

54. Hyperbola referred to axes parallel to its axes of symmetry

Reasoning as in the case of the ellipse, we easily deduce that

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \pm 1 \tag{32}$$

represents hyperbolas with center (h,k) and axes parallel to their axes of symmetry. Also, the equation

$$Ax^2 + By^2 + Cx + Dy + E = 0, \tag{33}$$

having A and B opposite in sign, can generally be changed to one of the forms (32) by completing the square on the terms in x and the terms in y and algebraic manipulation. The following example will illustrate the procedure.

Example. Sketch the hyperbola

$$9x^2 - 4y^2 - 36x - 8y = 4.$$

Solution. Completing the square on the terms in x and those in y and simplifying slightly, obtain

$$9(x^2 - 4x + 4) - 4(y^2 + 2y + 1) = 4 + 36 - 4;$$

$$\frac{(x-2)^2}{4} - \frac{(y+1)^2}{9} = 1. \quad \text{(a)}$$

From this, by means of the transformation

$$x = x' + 2, \; y = y' - 1, \quad \text{(b)}$$

obtain
$$\frac{x'^2}{4} - \frac{y'^2}{9} = 1.$$

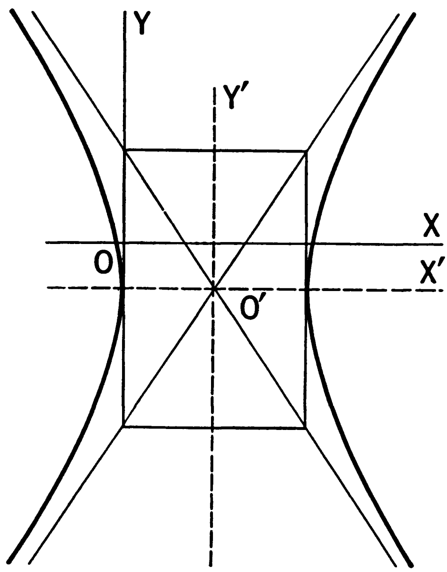


FIG. 39.

Figure 39 shows both sets of axes and the graph.

Exercises

Find the equations of the hyperbolas having the indicated centers and lengths of axes numbered 1 to 4:

1. Center, (1,3); transverse axis, 4 parallel to x-axis; conjugate axis, 6.

2. Center, (0,4); transverse axis, 4 parallel to y-axis; conjugate axis, 4.

3. Center, $(0,-3)$; transverse axis, 6 parallel to y-axis; conjugate axis, 10.

4. Center, $(0,k)$; transverse axis, n parallel to x-axis; conjugate axis, k.

Sketch each hyperbola defined by the equations numbered 5 to 8. Also give eccentricity and coordinates of vertices:

5. $x^2 + 2x - y^2 + 2y = 1.$ **7.** $3x^2 - y^2 + 12x - 24 = 0.$

6. $x^2 + 2x - y^2 - 4y - 4 = 0.$ **8.** $y^2 - 3x^2 + 4y = 0.$

For each set of data numbered 9 to 12, find the equation of the corresponding hyperbola:

9. Center, (1,3); foci, $(1 \pm 3,2)$; conjugate axis, 4.

10. Center, $(0,b)$; foci on y-axis; transverse axis, 8; $e = \frac{5}{4}$.

11. Vertices, $(3 \pm 2,3)$; $e = \sqrt{2}$; conjugate axis, 4.

12. Center, $(1,-1)$; one focus, $(1,-4)$; transverse axis, 3.

13. Find the equation of the system of hyperbolas with center on the y-axis, $e = 2$, and transverse axis 6.

14. Find the locus of a point having the difference of its distances from (2,1) and (7,1) equal to 3.

15. Find the locus of a point having the difference of its distances from $(-1,-2)$ and $(-1,6)$ equal to 6.

16. Find the equation of the locus of the midpoints of line segments perpendicular to the transverse axis prolonged of $x^2 - 4y^2 - 4x - 16y = 19$ and terminated by this prolonged axis and points on the hyperbola.

17. Taking a focus as pole and a line perpendicular to the directrix as polar axis (see Figure 40), show that a polar equation of a hyperbola is

$$\rho = \frac{pe}{1 + e\cos\theta}.$$

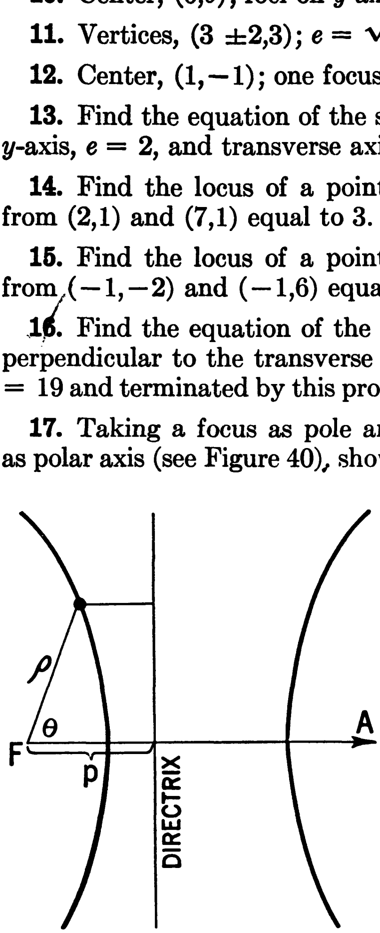


FIG. 40.

18. Find p and e for each hyperbola:

(a) $\rho(2 + 3\cos\theta) = 9.$

(b) $\rho(1 + 2\cos\theta) = 5.$

For the hyperbolas represented by the equations numbered 19 and 20, find the intercepts on the polar axis and on $\theta = 90°$ by letting θ equal 0°, 180°, 90°, and 270°, in order. Plot the corresponding points, sketch the hyperbola, and find the lengths of its axes.

19. $\rho(1 - 3\cos\theta) = 12.$ **20.** $\rho(2 - 5\cos\theta) = 105.$

21. Using the method of §37 and the equation of Exercise 17, show that the equation of a hyperbola with a focus F at the pole and with

its transverse axis along a directed line FA' which makes an angle α with the polar axis FA (see Figure 41) is

$$\rho = \frac{pe}{1 + e\cos(\theta - \alpha)}.$$

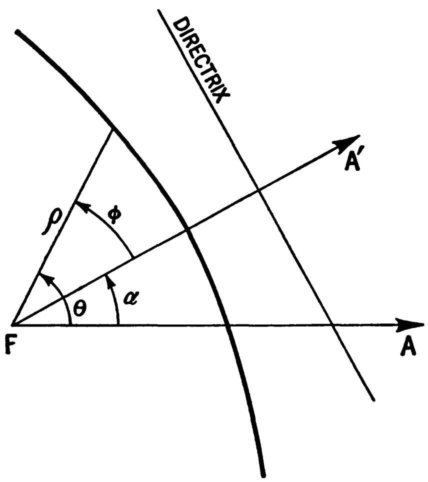


FIG. 41.

22. Using the equation in Exercise 21 with α equal to 0°, 180°, 90°, and 270° in turn, derive the following equations:

$$\rho(1 \pm e\cos\theta) = pe, \qquad \rho(1 \pm e\sin\theta) = pe,$$

and sketch the curves.

Find the intercepts on the lines $\theta = 0$ and $\theta = 90°$ of the graph of each of the equations numbered 23 to 26, sketch it, and find the lengths of its axes:

23. $\rho(1 + 2\sin\theta) = 12.$ **25.** $\rho(3 + 5\cos\theta) = 20.$

24. $\rho(3 - 4\sin\theta) = 28.$ **26.** $\rho[3 + 5\cos(\theta - 30°)] = 20.$

27. Find in rectangular form the equation of the hyperbola represented by

$$x = h + a\sec\phi,\ y = k + b\tan\phi.$$

55. Applications of hyperbolas

The main applications are based on the fact that a hyperbola is the locus of a point having distances of constant difference from two fixed points. To locate an enemy gun, two listening posts are set up; then, by finding the number of seconds' difference in time, $t_2 - t_1$, of hearing the sound of the gun at these

posts, and therefore the difference in distance, $1100(t_2 - t_1)$ feet, the gun is located on a hyperbola with the listening posts as foci. If three stations are used, the gun is located at the intersection of three hyperbolas. The same plan is applied in a system

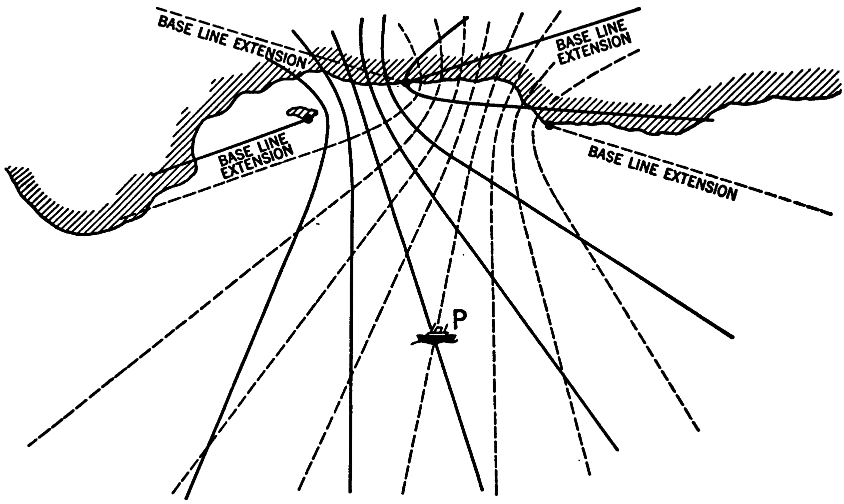


FIG. 42. The position *P* is located as the intersection of two hyperbolas determined by two time differences.

of navigation called **Loran,** which uses radio signals from selected stations, and maps with systems of hyperbolas, drawn in advance, having the stations as foci (see Figure 42). This system is very rapid and effective, especially since it is not affected by darkness, daylight, or weather and requires very little computation.

CHAPTER IX

General Equation of the Second Degree in x and y

56. Foreword

This chapter will be devoted to showing that the only kinds of curves representable by an equation having the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (1)$$

are parabolas, ellipses, hyperbolas, one or two straight lines, and a point. An equation having the form (1) may also represent no real locus at all. The method of attack consists in obtaining from the given equation by means of a transformation, considered in the next article, a new equation containing no term in xy. This new equation is easily treated by the methods of the last chapter.

57. Rotation of axes

Let the coordinates of any point P be x and y when referred to a set of coordinate axes, and let its coordinates be x' and y' when referred to another set of axes obtained by rotating the first set about the origin O through an angle α (see Figure 1). From P drop perpendiculars to OX and OX' meeting them in A and A', respectively. Then,

$$OA = x, AP = y, OA' = x', A'P = y'. \qquad (2)$$

Denoting angle AOP by θ, angle $A'OP$ by θ', and length OP by r, we have

$$x = r \cos \theta, \quad y = r \sin \theta, \qquad (3)$$

$$x' = r \cos \theta', y' = r \sin \theta', \qquad (4)$$

$$\theta = \theta' + \alpha. \qquad (5)$$

Substituting $\theta' + \alpha$ from (5) for θ in (3), we get

$$x = r\cos(\theta' + \alpha) = r\cos\theta'\cos\alpha - r\sin\theta'\sin\alpha, \tag{6}$$

$$y = r\sin(\theta' + \alpha) = r\sin\theta'\cos\alpha + r\cos\theta'\sin\alpha. \tag{7}$$

Replacing $r\cos\theta'$ and $r\sin\theta'$ in (6) and (7) by their values x' and y' from (4), we get

$$\begin{aligned} x &= x'\cos\alpha - y'\sin\alpha, \\ y &= x'\sin\alpha + y'\cos\alpha. \end{aligned} \tag{8}$$

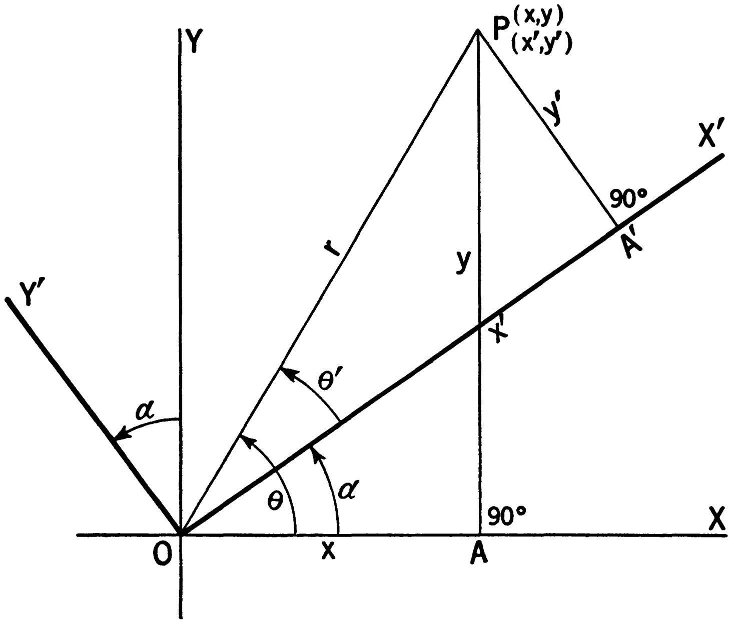


FIG. 1.

These are the fundamental equations for referring a curve to new axes making an angle α with the original axes. The solution of (8) for x' and y' in terms of x and y is

$$\begin{aligned} x' &= x\cos\alpha + y\sin\alpha, \\ y' &= -x\sin\alpha + y\cos\alpha. \end{aligned} \tag{9}$$

Both sets of equations, (8) and (9), are conveniently suggested by

$$\begin{array}{c|cc} & x' & y' \\ \hline x & \cos\alpha & -\sin\alpha \\ y & \sin\alpha & \cos\alpha \end{array} \tag{10}$$

The process of referring a curve, by means of (8), to a set of axes obtained by rotating the original set about the origin will be called **rotation of axes.**

The following example illustrates the use of the transformation (8) to obtain from a given equation having the form (1) a new one free of the term in xy.

Example. Apply rotation of axes through the angle $\tan^{-1}(2)$ to

$$17x^2 - 12xy + 8y^2 = 20. \tag{a}$$

Draw both sets of axes and the graph.

Solution. From Figure 2, for $\alpha = \tan^{-1}(2)$, find

$$\sin\alpha = 2/\sqrt{5},\ \cos\alpha = 1/\sqrt{5}. \tag{b}$$

Substituting these values in (8), obtain

$$x = \frac{x' - 2y'}{\sqrt{5}},\ y = \frac{2x' + y'}{\sqrt{5}}. \tag{c}$$

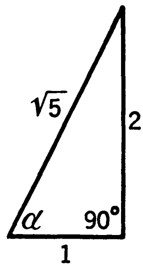


FIG. 2.

Substituting from (c) in (a), obtain

$$17\left(\frac{x' - 2y'}{\sqrt{5}}\right)^2 - 12\left(\frac{x' - 2y'}{\sqrt{5}}\right)\left(\frac{2x' + y'}{\sqrt{5}}\right) + 8\left(\frac{2x' + y'}{\sqrt{5}}\right)^2 = 20, \tag{d}$$

or, simplified,

$$\frac{x'^2}{4} + \frac{y'^2}{1} = 1. \tag{e}$$

Figure 3 shows both sets of axes and the graph.

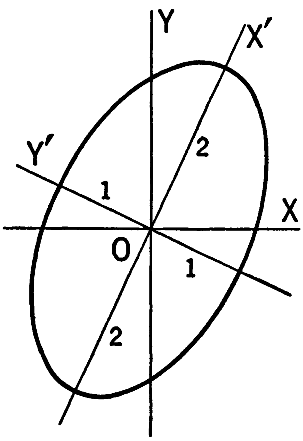


FIG. 3.

Exercises

Perform the indicated rotations of axes in the exercises numbered 1 to 10, draw both sets of axes, and sketch the curves from the simplified equations:

1. $xy = 4;\ \alpha = 45°$.

2. $13x^2 - 18xy + 37y^2 = 160;\ \alpha = \tan^{-1}\frac{1}{3}$.

3. $25x^2 + 6\sqrt{3}xy + 19y^2 = 16;\ \alpha = 30°$.

4. $x^2 - 4xy + 4y^2 = 8\sqrt{5}y - 4\sqrt{5}x;\ \alpha = \tan^{-1}\frac{1}{2}$.

5. $5y^2 + 24xy - 5x^2 = 234;\ \alpha = \tan^{-1} \frac{3}{2}$.
6. $9x^2 + 24xy + 16y^2 = 30y - 40x;\ \alpha = \tan^{-1} \frac{4}{3}$.
7. $5x^2 - 8xy + 5y^2 = 9;\ \alpha = 45°$.
8. $5x^2 + 6\sqrt{3}xy - y^2 = -8;\ \alpha = 30°$.
9. $3x^2 - 10xy + 3y^2 = 3\sqrt{2}x - \sqrt{2}y;\ \alpha = 45°$.
10. $9x^2 + 4xy + 6y^2 - 8\sqrt{5}x + 4\sqrt{5}y + 2 = 0;\ \alpha = \tan^{-1} \frac{1}{2}$.
11. In Figure 4,

$$x = OA = OB - AB = OB - DC,$$
$$y = AP = AD + DP = BC + DP.$$

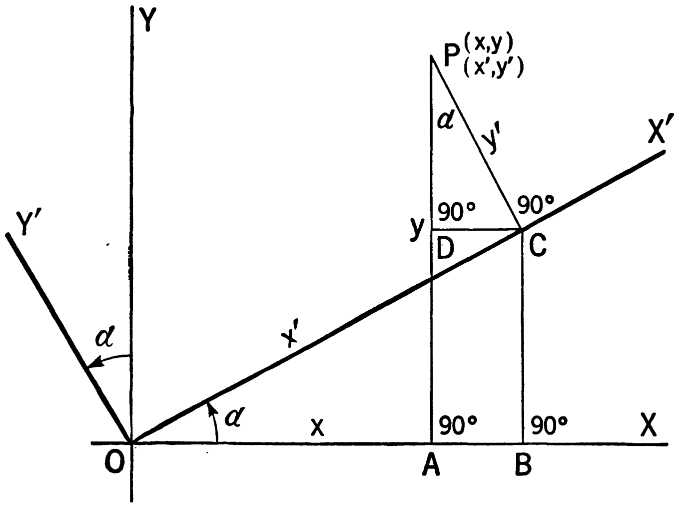


FIG. 4.

Replace OB, DC, BC, and DP in these equations by their values in terms of x', y', and α, from Figure 4, to obtain Equations (**8**). Also obtain Equations (**9**) directly from the figure by using the same type of argument.

12. Refer the line $x' = p$ to x- and y-axes by means of (**9**) and obtain the normal form of the equation of a straight line:

$$x \cos \alpha + y \sin \alpha = p.$$

58. Reduction of second-degree equations to standard forms

In §**57**, equations were simplified by rotation of axes. In this article the general plan for reduction to standard form will be considered. Replacing x and y by their values from Equations (**8**) in

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (11)$$

and collecting coefficients of like terms, we obtain a new equation,

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F = 0, \qquad (12)$$

where*

$$A' = A\cos^2\alpha + B\sin\alpha\cos\alpha + C\sin^2\alpha, \qquad (13)$$

$$\begin{aligned} B' &= 2(C - A)\sin\alpha\cos\alpha + B(\cos^2\alpha - \sin^2\alpha) \\ &= (C - A)\sin 2\alpha + B\cos 2\alpha,\dagger \end{aligned} \qquad (14)$$

$$C' = A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha, \qquad (15)$$

$$\left.\begin{aligned} D' &= D\cos\alpha + E\sin\alpha, \\ E' &= -D\sin\alpha + E\cos\alpha, \\ F' &= F. \end{aligned}\right\} \qquad (16)$$

We can eliminate the term in $x'y'$ in (12) by equating B' from (14) to zero, obtaining

$$B' = (C - A)\sin 2\alpha + B\cos 2\alpha = 0, \qquad (17)$$

or, dividing through by $\cos 2\alpha$ and solving for $\tan 2\alpha$,

$$\tan 2\alpha = \frac{B}{A - C}, \quad A - C \neq 0. \qquad (18)$$

If $A = C$, $\tan 2\alpha$ is undefined. Then, however, from (17), $B' = B\cos 2\alpha = 0$, which is satisfied by $2\alpha = 90°$ or $\alpha = 45°$.

Rotation of axes through angle α defined by (18), *when applied to a second-degree equation having the form* (11), *gives a new equation with the* x′y′*-term missing.*

In using (18) for any particular case, it is convenient to take 2α less than 90° when $B/(A - C)$ is positive and to take 2α between 90° and 180° when $B/(A - C)$ is negative. Then α is always less than 90°, $\sin\alpha$ and $\cos\alpha$ are both positive, and from trigonometry

$$\sin\alpha = \sqrt{\frac{1 - \cos 2\alpha}{2}}, \quad \cos\alpha = \sqrt{\frac{1 + \cos 2\alpha}{2}}. \qquad (19)§$$

* The student should carry out the transformation and check Formulas (13) to (16).

† From trigonometry, $\sin 2\alpha = 2\sin\alpha\cos\alpha$, $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$.

§ See Formulas 31 and 32, page 259.

Hence, *to transform a second-degree equation having the form* (11) *to a new equation with* x′y′*-term missing, use* (18) *to find* tan 2α, *then find* cos 2α, *taking it positive or negative according as* tan 2α *is positive or negative, substitute this value of* cos 2α *in* (19), *substitute the resulting values from* (19) *in* (8), *and carry out the corresponding transformation. The equation with* x′y′*-term missing may then be dealt with by the methods of the last chapter.*

Example 1. Transform the equation

$$2x^2 - 3xy - 2y^2 + 1.5\sqrt{10}x + 4.5\sqrt{10}y - 20 = 0 \qquad \text{(a)}$$

to standard form and sketch the curve represented by it.

Solution. From (18),

$$\tan 2\alpha = \frac{B}{A - C} = \frac{-3}{2 - (-2)} = \frac{-3}{4}. \qquad \text{(b)}$$

Hence, $\cos 2\alpha = -\frac{4}{5}$. Using this in (19), obtain $\sin \alpha = 3/\sqrt{10}$, $\cos \alpha = 1/\sqrt{10}$. Using these in (8), obtain

$$x = (x' - 3y')/\sqrt{10}, y = (3x' + y')/\sqrt{10}. \qquad \text{(c)}$$

Substituting these values in (a), and simplifying slightly, obtain

$$x'^2 - y'^2 - 6x' + 8 = 0. \qquad \text{(d)}$$

Then, by translation of axes with new origin (3,0), obtain

$$x''^2 - y''^2 = 1.$$

The graph is easily plotted on the last set of coordinate axes. Figure 5 shows all three sets of coordinate axes and the required graph.

59. Discussion of the general equation of the second degree. Invariants

From (13) and (15), §58, we easily obtain

$$A + C = A' + C' \qquad (20)$$

for the general equations (11) and (12). Because of (20), the quantity $A + C$ is called an **invariant** under rotation of axes. Invariants are very useful in many types of investigations. By using equations (13), (14), and (15), the student may verify that

$$B^2 - 4AC = B'^2 - 4A'C'. \qquad (21)$$

This invariant will play a prominent role in the following discussion relating to the types of graphs represented by equations having the general quadratic form.

Application of the rotation of axes transformation to (11) for an angle α defined by $\tan 2\alpha = B/(A - C)$ gives a new equation,

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F = 0, \qquad (22)$$

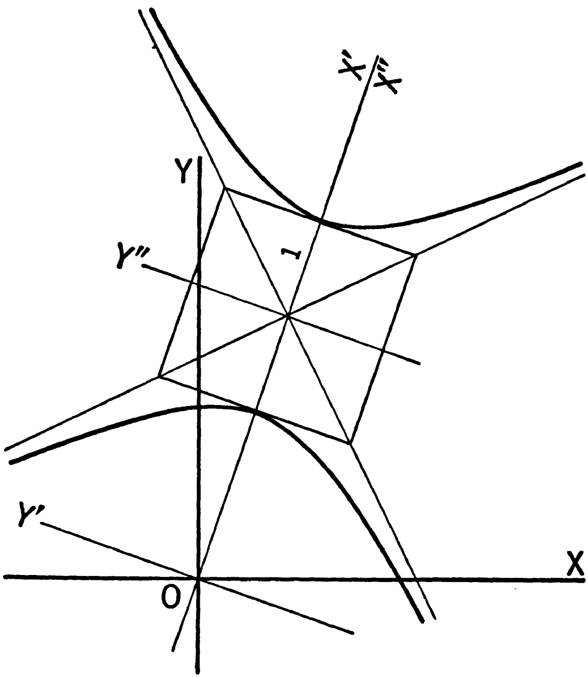


FIG. 5.

with the $x'y'$-term missing. By using the methods of Chapter VIII, we may easily show that (22) may represent:

An **ellipse,** *no locus, or a point, if* A′ *and* C′ *have like signs,* (A)

A **hyperbola** *or two intersecting lines if* A′ *and* C′ *have opposite signs,* (B)

A **parabola,** *no locus, a line, or two parallel lines, if either* $A' = 0$ *or* $C' = 0$. (C)

Hence, equations having the form (22), and, therefore, equations having the form (11), may represent no locus, a point, a line, two lines, an ellipse, a parabola, or a hyperbola. Now (21) with $B' = 0$ becomes

$$B^2 - 4AC = -4A'C'. \qquad (23)$$

Considering this relation and relations (A), (B), and (C), it appears that the general equation (11) represents

An **ellipse,** *no locus, or a point if* $\mathbf{B^2 - 4AC < 0}$, (D)

A **hyperbola** *or two lines if* $\mathbf{B^2 - 4AC > 0}$, (E)

A **parabola,** *no locus, one line, or two lines if* $\mathbf{B^2 - 4AC = 0}$. (F)

Exercises

By using (8), (18), and (19), write the transformation to be used in deriving from each of the equations numbered 1 to 6 a new equation having no term in $x'y'$:

1. $9x^2 + 24xy + 2y^2 = 10$. **4.** $x^2 - \sqrt{3}xy + 2y^2 = 5$.

2. $25x^2 - 7xy + y^2 = 8$. **5.** $x^2 - 2\sqrt{2}xy + 2y^2 = 12$.

3. $x^2 + 2xy + y^2 = 12x$. **6.** $4xy = 6$.

7. Assuming that each of the equations numbered (1) to (6) represents an ellipse, a parabola, or a hyperbola, identify the graphs represented by them by using the expressions marked (D), (E), and (F).

By rotation of axes find from each of the equations numbered 8 to 10 a new equation with no term in $x'y'$. In each case check that the corresponding Equations (20) and (21) hold true:

8. $x^2 + xy + y^2 - 4\sqrt{2}x + 4\sqrt{2}y = 10$.

9. $4x^2 + 6xy - 4y^2 + x + 7y = 350$.

10. $x^2 + 4xy + 4y^2 = \sqrt{5}(2x + y)$.

By rotation and translation of axes, obtain equations in standard form from the equations given in the exercises numbered 11 to 19. Plot all sets of axes used and the graph of each equation.

11. $x^2 + xy + y^2 = 4\sqrt{2}x - 4\sqrt{2}y + 10$.

12. $4x^2 + 6xy - 4y^2 = 5$.

13. $x^2 + 4xy + 4y^2 = \sqrt{5}(2x + y)$.

14. $2xy + \sqrt{2}x + \sqrt{2}y = 3$.

15. $13x^2 + 6\sqrt{3}xy + 7y^2 = 16$.

16. $2x^2 + 2xy + 2y^2 - 10x + 4y + 13 = 0$.

17. $7x^2 + 8xy + y^2 + 81 = 0$.

18. $x^2 + 2xy + y^2 - \sqrt{2}x + \sqrt{2}y = 0$.

19. $xy + 4x + 3y + 4 = 0$.

20. Find the equation of the locus of a point so situated that the ratio of its distance from the origin to its distance from $x + y = 2\sqrt{2}$ is

always 1. By rotation and translation of axes applied to the result, find a standard form of the locus and sketch it.

21. In Exercise (20) replace the ratio 1 by $\frac{1}{2}$, and solve the result.

22. In Exercise (20) replace the ratio 1 by $\sqrt{2}$, and solve the result.

23. Prove that $Ax^2 + Bxy + Cy^2 = K$ cannot represent an ellipse if A and C have opposite signs. Can it represent a hyperbola if A and C have the same sign?

24. Prove that neither $Ax^2 + Bxy = D$ nor $Bxy + Cy^2 = D$ can represent an ellipse or a parabola.

25. Show that $2xy - \sqrt{2}(x - y) = 1$ represents two straight lines.

★26. Show that $41x^2 + 24xy + 34y^2 + 30x - 40y + 25 = 0$ represents a point.

★27. Show analytically that the distance between two points is invariant under rotation; that is, show that

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2}$$

if the relation (8) holds true.

CHAPTER X

Miscellaneous Curves

60. Foreword

This chapter deals with various curves not already discussed and with curves which are of considerable importance in higher mathematics. Many of these resemble the hyperbola in that they have one or more asymptotes. For the most part, each asymptote considered will be parallel to one of the coordinate axes. Illustrative of these curves are the graphs of the trigonometric functions, the logarithmic function, and exponential functions. A good example of a curve of practical importance is the curve showing characteristics of oscillatory motion slowed down, or damped, by friction or resistance,

$$y = e^{-ax} \sin (bx + c). \tag{1}$$

61. Curves having asymptotes parallel to the coordinate axes

Asymptotes were considered in §**13.** Here the discussion will be amplified and more complicated graphs considered.

If as x *takes on values nearer and nearer without limit to a value* a, y = f(x) *takes on values which become numerically greater and greater without bound, then* x = a *is an* **asymptote** *of* y = f(x). *Also, the statement holds if in it* x *and* y *are interchanged.**

A function $f(x)/(x - a)$, where $f(a)$ is not zero, becomes great without limit as x approaches a, just as does $1/(x - a)$ when x approaches a. For, when x is near a, $f(x)$ is nearly equal to $f(a)$ and $x - a$ becomes indefinitely small. Accordingly, $x - a = 0$ is an asymptote of $y = f(x)/(x - a)$. This fact is

* A more general definition of an asymptote follows: If as a point moves out on a curve farther and farther without bound, the distance from the point to a line becomes and remains less than any positive value that can be preassigned, then the line is an asymptote of the curve.

the basis of the following statements relating to asymptotes parallel to the axes.

If y *is equal to a fraction*

$$y = \mathrm{N}(\mathrm{x})/\mathrm{D}(\mathrm{x}), \tag{2}$$

where the numerator N(x) *and the denominator* D(x) *are not both zero for any value of* x, *the equations of asymptotes parallel to the* y*-axis are obtained by setting* D(x) *equal to zero and solving for* x. Similarly, *if an equation can be solved for* x *as a fraction in terms of* y, *the equations of the asymptotes parallel to the* x*-axis may be found by setting the denominator of the result equal to zero and solving for* y.

When $N(x)$ and $D(x)$ in (2) are polynomials of the same degree, an asymptote parallel to the x-axis is generally obtained by deleting all but the highest powers of x. Thus, the equation of an asymptote of

$$y = \frac{2x^2 - 4x - 6}{x^2 + 2x - 3} \tag{3}$$

is $y = 2x^2/x^2 = 2$. For (3) may be written

$$y = \frac{2(x^2/x^2) - 4(x/x^2) - (6/x^2)}{(x^2/x^2) + 2(x/x^2) - (3/x^2)} = \frac{2 - (4/x) - (6/x^2)}{1 + (2/x) - (3/x^2)}, \tag{4}$$

and we see from this that the terms with x in the denominator are very small when x is very large, and that y approaches 2. Also, if the degree of $D(x)$ is higher than that of $N(x)$, $y = 0$ is an asymptote of (2). Thus, if

$$y = \frac{x}{x^2 + 4} = \frac{1}{x + \dfrac{4}{x}},$$

y approaches zero as x increases without limit, and $y = 0$ is an asymptote.*

In graphing a curve, all data easily obtained relative to intercepts, symmetry, and extent should be obtained. Note that the

* A good working rule follows: To find the equations of asymptotes parallel to the axes for an equation having only terms of the form ax^my^n, where m and n are non-negative rational numbers, equate to zero the linear factors of the coefficient of the highest power of y in the equation and of the highest power of x in the equation. Thus, the asymptotes parallel to the axes of $y^3(x^2 - 1) + x^3(y + 3) = 0$ are given by

$$x^2 - 1 = 0,\ y + 3 = 0;\ \text{or}\ x = 1,\ x = -1,\ y = -3.$$

intercepts on the x-axis are obtained from (2) by setting $N(x) = 0$ and solving for x. Thus, the x-intercepts of

$$y = \frac{(x-1)(x+2)^2}{x-3} \tag{5}$$

are found by writing

$$(x-1)(x+2)^2 = 0, \text{ or } x = 1, x = -2. \tag{6}$$

When a factor $(x - a)$ in the numerator $N(x)$ of (2) appears to an even degree, the graph meets the x-axis at $(a,0)$ but does not cross the x-axis, since the sign of y is the same for all values of x near a. Hence, the graph is generally tangent to the x-axis at $x = a$. Thus, the graph of (5) is tangent to the x-axis at $x = -2$. If a factor $(x - a)$ occurs to an odd power greater than 1 in $N(x)$, the graph of (2) will be tangent to the x-axis and cross it. Figure 1

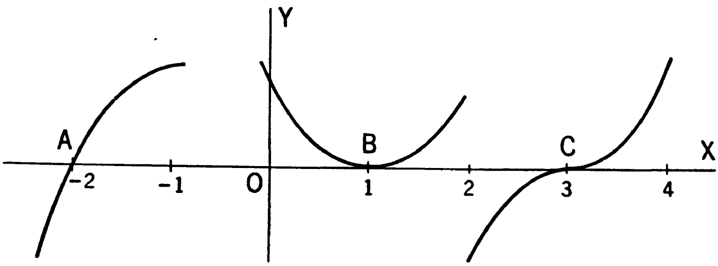


FIG. 1.

shows typical graphs for various kinds of factors of $N(x)$: Point A relates to a factor $(x + 2)$ of N, point B to a factor $(x - 1)^2$ of N, and point C to a factor $(x - 3)^3$ of N.

It should also be mentioned that, even for a sketch of a curve, a table giving coordinates of a set of points on a curve is valuable to check results found by the given analysis.

Example. Sketch the graph of

$$y = (x-2)(x+2)^2/[x^2(x+4)].$$

Solution. The x-intercepts are given by

$$(x-2)(x+2)^2 = 0, \text{ or } x = 2, x = -2.$$

At $x = -2$, the x-axis is tangent to the curve. Investigation for symmetry and extent yields no useful information. The equations of the asymptotes parallel to the y-axis are given by

$$(x+4)\,x^2 = 0, \text{ or } x = -4, x = 0.$$

If x is very large, we have, approximately,

$$y = x(x^2)/[x(x^2)] = 1,$$

so that $y = 1$ is an asymptote. Using the facts mentioned and the following table showing corresponding values of x and y,

x	-8	-6	-5	-3	-2	-1	1	2	3	5
y	1.4	1.8	2.5	$-.56$	0	-1	-1.8	0	$.40$	$.65$

we plot the graph in Figure 2.

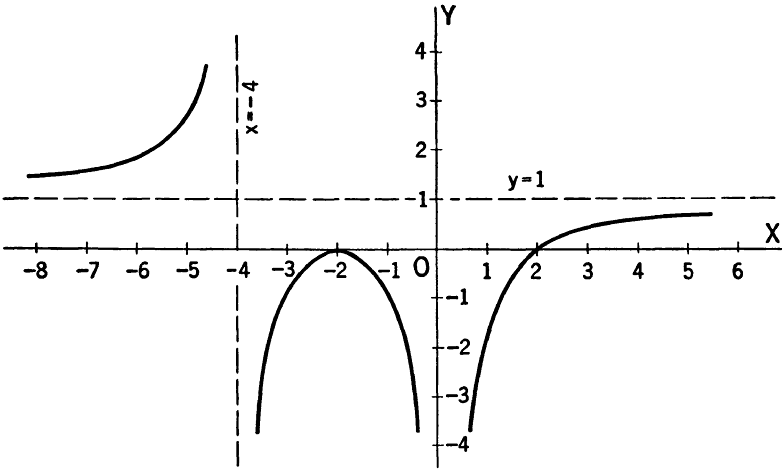


FIG. 2.

Exercises

Find the equations of asymptotes parallel to the axes for the equations numbered 1 to 4:

1. $y(x - 2) = 4.$
2. $(x^2 - 1)y = x^2 + 1.$
3. $xy = 5.$
4. $(x^2 - 1)y^3 + (y^2 - 4)x^3 = x + 3.$

Sketch the graphs of the following equations:

5. $x(y - 2) = 2.$
6. $(x^2 - 1)y = 3.$
7. $(x^2 - 1)y = x^2 + 1.$
8. $x^2y = x^2 - 4.$
9. $y(x + 2) = (x - 2)^2.$
10. $y(x - 2)^2 = x + 2.$
11. $y(x^2 + 4) = 4x.$
12. $y(x^2 - 4) = 4x.$

13. $y(x^2 - 16) = x^2$.

14. $x(y^2 - 6y + 8) = y^2$.

15. $x(2y - 3) + 6y + 4 = 0$.

16. $(x^2 - 4)y = x^2 - 9$.

17. $x^2y + 4x + y = 2$.

18. $y = (x - 3)^2/(x^2 - 4)$.

Translate the origin to the indicated point and sketch the graphs of the equations numbered 19–22:

19. $(x - 2)(y - 3) = 4$; (2,3).

20. $x^2(y - 3) = 2$; (0,3).

21. $(x - 1)^2(y + 2) = 2$; (1,−2).

22. $(x^2 - 1)(y + 1) = (x - 1)^2 + 1$; (1,−1).

23. To find an oblique asymptote of $y = 2x^2/(x + 1)$, obtain by division

$$y = \frac{2x^2}{x+1} = 2x - 2 + \frac{2}{x+1},$$

and observe that, as x becomes great without limit, y approaches $2x - 2$; therefore, $y = 2x - 2$ is an asymptote. Use this asymptote as an aid in sketching the graph of the given equation. (Note that the curve is a hyperbola.)

For each of the equations numbered 24 to 27, find an oblique asymptote by means of the principle given in Exercise 23, and then sketch the graph of each equation.

24. $y(x - 1) = x^2 + 3$.

25. $y(x + 2) = (x - 1)(x - 3)$.

26. $4xy = 8x^2 + 2x - 1$.

27. $x^2y = x^3 - 1$.

28. Sketch on the same graph: $y = x$, $y = x^2$, $y = x^3$, $y = x^{-1}$, $y = x^{-2}$, $y = x^{-3}$.

29. Sketch on the same graph: $y^2 = x$, $y^2 = x^2$, $y^2 = x^3$, $y^2 = x^{-1}$, $y^2 = x^{-2}$.

62. Graphs of the trigonometric functions

In the following discussion angles will be expressed in radians. From the review of trigonometry, pages 256–259,

$$1° = \pi/180 \text{ radian} = 0.0175 \text{ radian, approximately,} \qquad (7)$$

$$1 \text{ radian} = (180/\pi)° = 57.3°, \text{ approximately.} \qquad (8)$$

By using the table on pages 260 and 261 to find desired values of the sine function and proceeding in the usual manner, one may plot

$$y = \sin x. \qquad (9)$$

Figure 3 shows the graph for the range of x, 0 to 2π.

Since $\sin x = \sin (x + 2\pi)$, the values of the sine are repeated at intervals of 2π, and therefore the graph is repeated, as in-

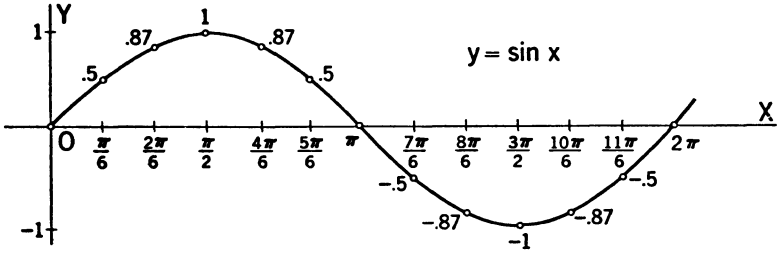


FIG. 3.

dicated in Figure 4. Because of the repetitive feature defined by $\sin x = \sin (x + 2\pi)$, *sin* x *is said to be periodic and to have the* **period** 2π.

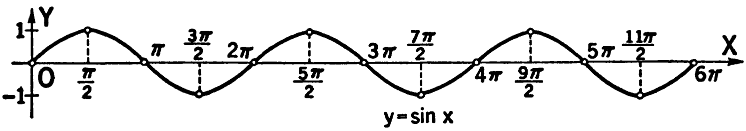


FIG. 4.

Next consider

$$y = a \sin kx. \tag{10}$$

From the table of values,

x	0	$\pi/(2k)$	$2\pi/(2k)$	$3\pi/(2k)$	$4\pi/(2k)$
$y = a \sin kx$	0	a	0	$-a$	0

we obtain the sketch of $y = a \sin kx$ shown in Figure 5. The length a shown in the figure is called the **amplitude** of the func-

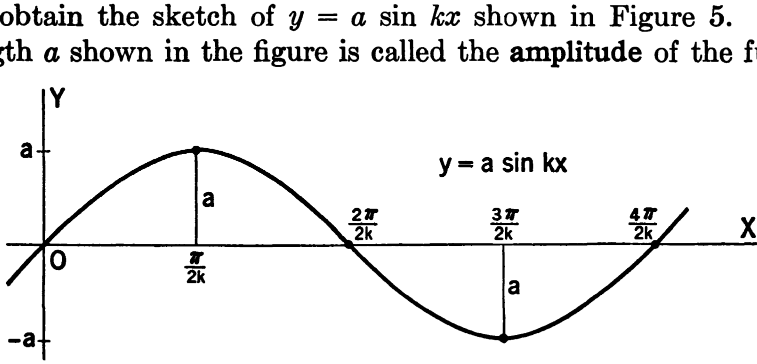


FIG. 5.

tion (10) and the variation $2\pi/k$ of x for a complete cycle of the curve is called its **period.**

The role played by the constants a and k in the graph of an equation having the form (10) is illustrated in Figure 6. The

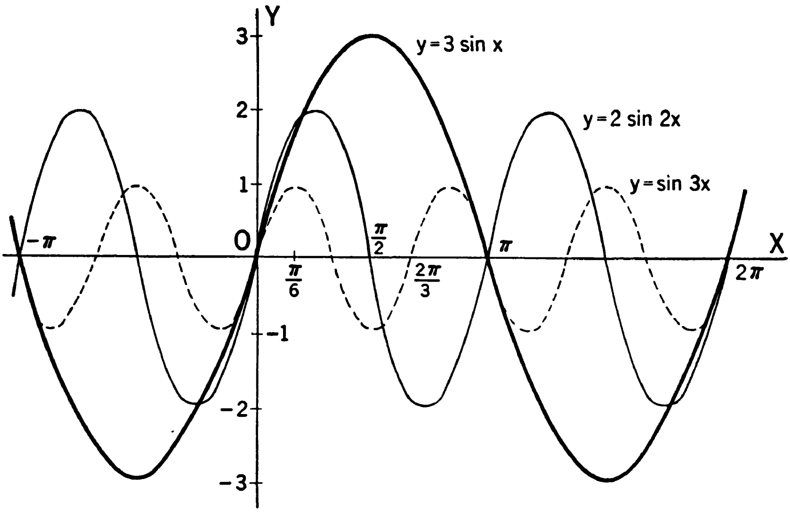


FIG. 6.

heavily drawn curve represents $y = 3 \sin x$, the **light** curve $y = 2 \sin 2x$, and the dotted curve $y = \sin 3x$. Note especially their amplitudes and periods.

To find the graph of

$$y = a \cos kx, \tag{11}$$

make the translation $x = x' - \pi/(2k)$, $y = y'$ to obtain

$$y' = a \cos (kx' - \pi/2) = a \sin kx'. \tag{12}$$

Hence, the graph of the cosine curve (11) is the same as the graph of the sine curve (10) referred to axes moved $\pi/(2k)$ to the left. Figure 7 shows both sets of axes and the curve (11). Observe that functions $a \sin kx$ and $a \cos kx$ have the same shape, period, and amplitude.

As x takes on values indefinitely near to 90°, $y = \tan x$ takes on values numerically great without bound. Hence, the graph of $y = \tan x$ has as asymptote $x = \pi/2$; and the same reasoning shows that $x = (2n + 1)\pi/2$ is an asymptote provided n is an integer. Moreover,

$$\tan x = \tan (\pi + x),$$

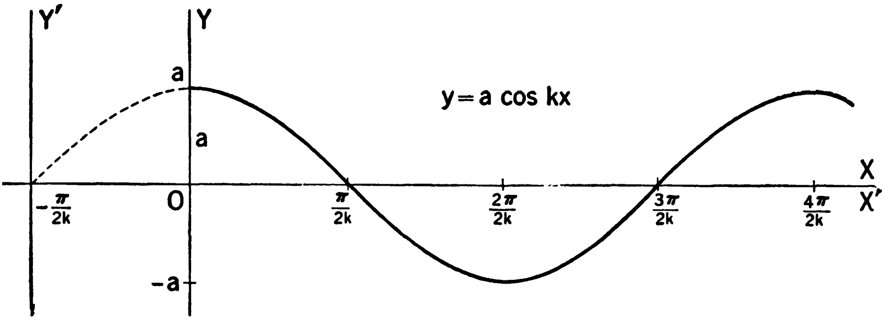


FIG. 7.

and therefore tan x has the period π. Also,

$$\tan kx = \tan k(x + \pi/k), \tag{13}$$

and therefore **tan kx has π/k as period.** Figure 8 shows part of the graph of $y = \tan x$.

A sketch of $y = \csc x$ is easily obtained from that of $y = \sin x$ by using the relation $\csc x = 1/\sin x$. Since $\sin 0 = 0$, $\sin (\pm\pi)$

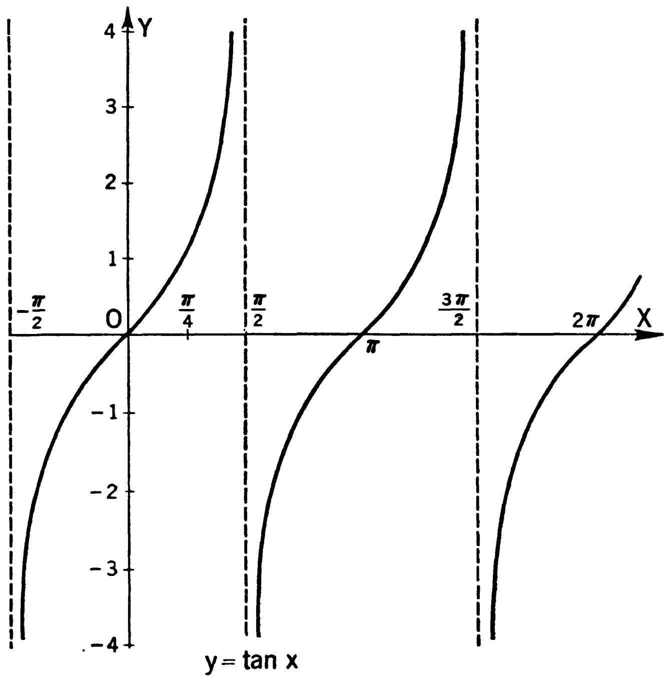


FIG. 8.

$= 0$, $\sin(\pm 2\pi) = 0, \cdots$, the graph of $y = \csc x$ will have as asymptotes $x = 0, x = \pm\pi, x = \pm 2\pi, \cdots$. When $\sin x = \pm 1$, $\csc x = \pm 1$; hence, the corresponding points on $y = \sin x$ and

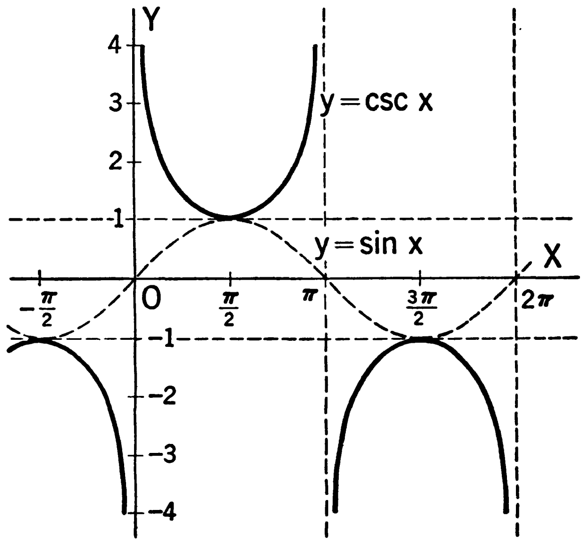


FIG. 9.

$y = \csc x$ will be the same. For other values of x, $\csc x$ may be estimated by using $\csc x = 1/\sin x$. Figure 9 shows, in part, the graph of $y = \sin x$ dotted and $y = \csc x$ in full line. Of course, the graph could be obtained by direct plotting.

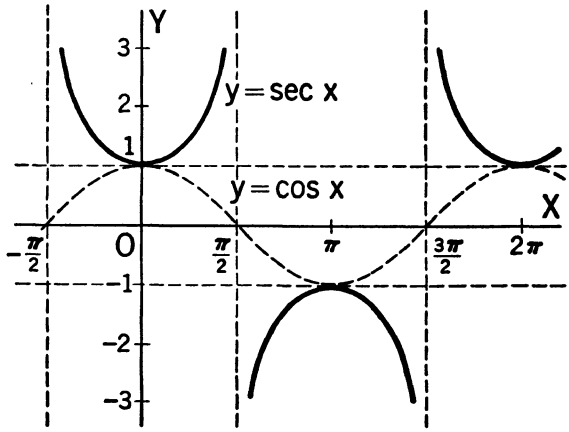


FIG. 10.

Similarly, $\sec x = 1/\cos x$ and $\cot x = 1/\tan x$; hence, the graphs of $\sec x$ and $\cot x$ can be obtained from those of $y = \cos x$ and $y = \tan x$. Figures 10 and 11 show these graphs.

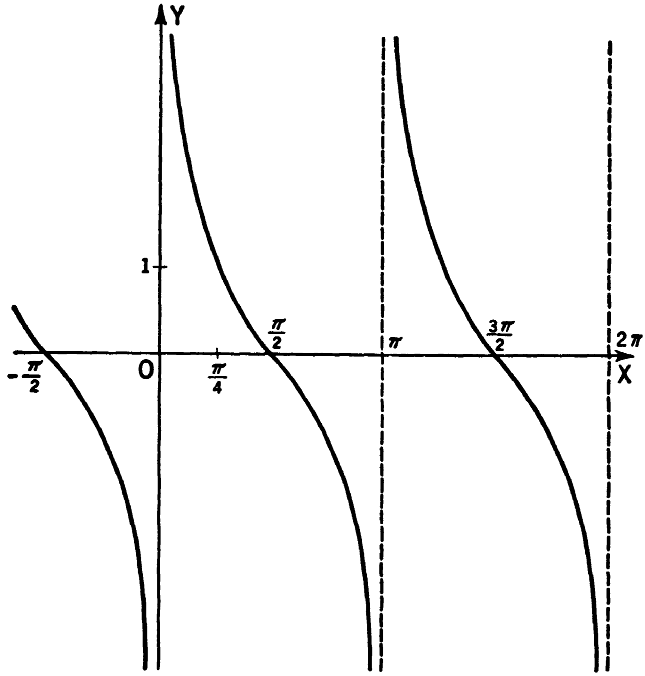


FIG. 11.

Exercises

Give the period and the amplitude, if it has one, of each of the functions numbered 1 to 6:

1. $y = 3 \sin 2x$.

2. $y = 4 \cos 3x$.

3. $y = 6 \tan 3x$.

4. $y = 2 \cot 2x$.

5. $y = 3 \csc 3x$.

6. $y = 4 \sec 5x$.

Sketch the graph of each of the functions numbered 7 to 16:

7. $y = 3 \sin x$.

8. $y = 3 \cos x$.

9. $y = 2 \sin 2x$.

10. $y = 2 \cos 2x$.

11. $y = \tan x$.

12. $y = \tan 2x$.

13. $y = \cot 2x$.

14. $y = \sec 2x$.

15. $y = 2 \csc 2x$.

16. $y = \sin 3x$.

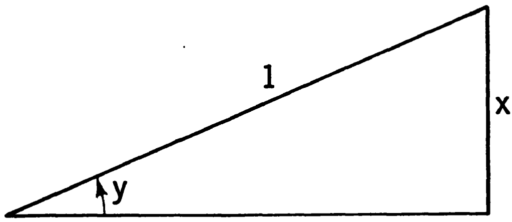


FIG. 12.

For each curve represented by one of the equations numbered 17 to 20, make the indicated translation and sketch the curve relative to the new axes:

17. $y = 2 \sin (x - \pi/4)$; $(\pi/4,0)$.

18. $y = 3 \cos (x + \pi/3)$; $(- \pi/3,0)$.

19. $y = \cos 2 (x - \pi/6)$; $(\pi/6,0)$.

20. $y = 2 \sin (2x + \pi/3)$; $(- \pi/6,0)$.

63. Graphs of the inverse trigonometric functions

Either of the equations

$$\mathbf{y = sin^{-1} x, y = arc \sin x} \tag{14}$$

means that y *is the angle whose sine is* x. Figure 12 shows the relation for y an angle of the first quadrant. From (14),

$$\sin y = x. \tag{15}$$

Since this is Equation (9), §62, with x and y interchanged, Equation (15), and therefore Equation (14), have graphs of the same shape as that of $y = \sin x$ but placed relative to the y-axis as $y = \sin x$ is placed relative to the x-axis. Since points (a,b) and (b,a) are symmetric with respect to line $y = x$, it follows that $y = \sin x$ and $x = \sin y$ are symmetric to each other with respect to $y = x$. Figure 13 shows the graph of (15) and, therefore, of (14). To each value of x correspond infinitely many values of y. Thus, the line $x = \frac{1}{2}$ cuts the graph of Figure 13 in points $(\frac{1}{2}, \frac{1}{6}\pi)$, $(\frac{1}{2}, \frac{5}{6}\pi)$, $(\frac{1}{2}, \frac{13}{6}\pi)$, $\cdots$. *We define the* **principal value** *of* y = sin^{-1} x *to be the one which is* $\pi/2$, $-\pi/2$, *or some value between them.* The heavily marked part of the graph has reference to the principal value. Unless otherwise indicated, we shall understand by the symbol $y = \sin^{-1} x$ the principal value.

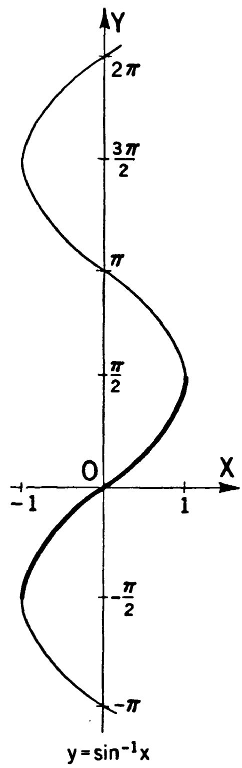


FIG. 13.

Similarly, the symbols

$$y = \tan^{-1} x, \text{ and } y = \text{arc} \tan x \tag{16}$$

will mean the angle y between $-\pi/2$ and $\pi/2$ having x as tangent. From (16), we have

$$x = \tan y. \tag{17}$$

The graph of (17) has the same shape as that of $y = \tan x$ but it is placed relative to the y-axis as $y = \tan x$ is placed relative to the x-axis. Figure (14) shows the graph of (17). The princi-

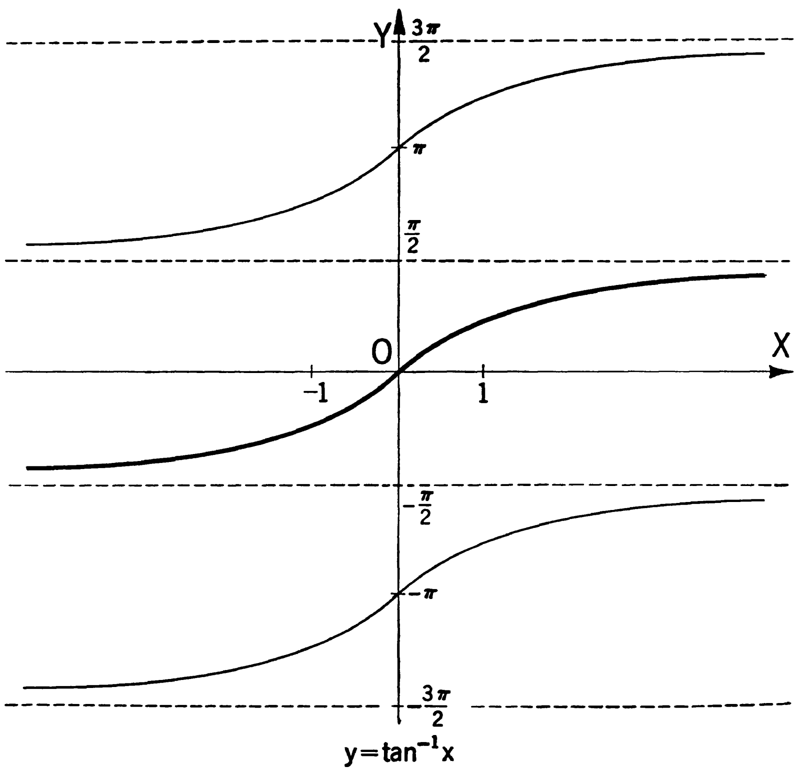


FIG. 14.

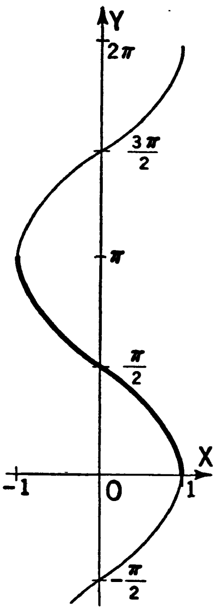
pal value of $\tan^{-1} x$ lies between $-\pi/2$ and $\pi/2$. The heavily marked part of Figure 14 has reference to principal values.

Similar remarks apply to the symbols and meanings of

$$y = \cos^{-1} x, \text{ and } y = \text{arc} \cos x, \tag{18}$$

$$y = \cot^{-1} x, \text{ and } y = \text{arc} \cot x. \tag{19}$$

The range of the values in this case is from 0 *to* π. Figures 15 and 16 show the respective graphs of $x = \cos y$ and $x = \cot y$ with the parts relating to (18) and (19) heavily marked.



$y = \cos^{-1} x$

FIG. 15.

Exercises

1. Give the range of the principal values of: (a) $\sin^{-1} x$. (b) $\cos^{-1} x$. (c) $\tan^{-1} x$. (d) $\cot^{-1} 2x$.

Sketch the graph of each of the functions numbered 2 to 7 (for principal values only):

2. $y = 2 \sin^{-1} x$.
3. $y = 2 \tan^{-1} x$.
4. $y = 2 \cos^{-1} x$.
5. $y = \sin^{-1} 2x$.
6. $y = 2 \cos^{-1} 2x$.
7. $y = 3 \tan^{-1} 2x$.

Sketch the graphs of the functions numbered 8 and 9. Assume the range for $\sec^{-1} x$ and $\csc^{-1} x$ to be from $-\pi$ to $-\pi/2$, and from 0 to $\pi/2$.

8. $y = \sec^{-1} x$.
9. $y = \csc^{-1} x$.

For each curve represented by the equations numbered 10 to 13, make the indicated translation and sketch:

10. $y = \sin^{-1} (x - 2)$; $(2,0)$.
11. $y = \cos^{-1} (x + 2)$; $(-2,0)$.
12. $y = \tan^{-1} 2 (x - 3)$; $(3,0)$.
13. $y = 2 \sin^{-1} (3x + 4)$; $(-\frac{4}{3},0)$.

64. Graphical addition of curves

If a function can be written as the sum of two or more functions, each easily plotted, then the graph of the original function can be obtained by adding the ordinates of the graphs of the component functions.

Consider, for example,

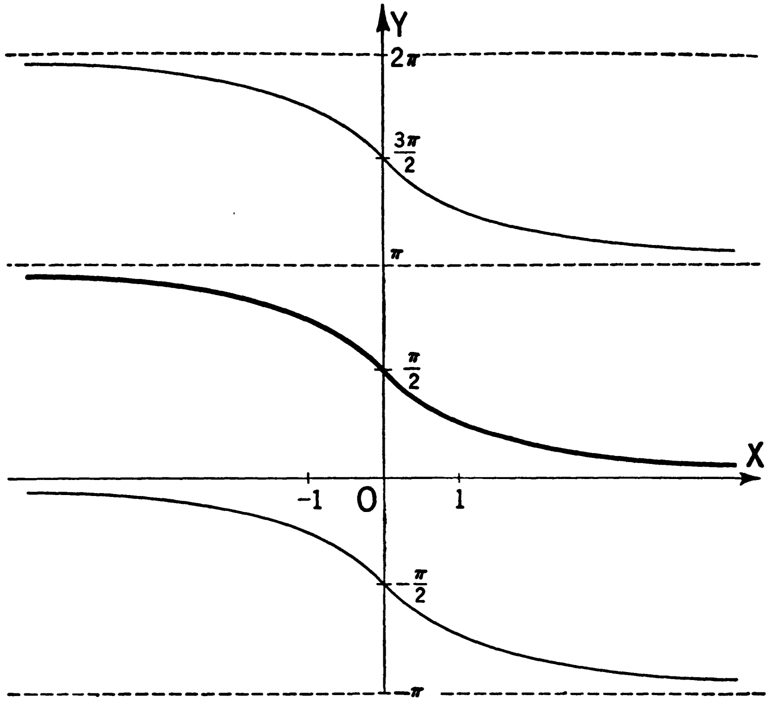
$$y = 2 \sin x + \cos 2x. \tag{20}$$

In Figure 17, $y = \cos 2x$ is dotted, $y = 2 \sin x$ is drawn lightly, and the graph of (20) obtained from the others by addition of ordinates is drawn in heavily. D represents any point on the graph of (20); observe that D was located by adding $CD = AB$ to AC, presumably by means of dividers. Similarly, H was obtained by subtracting GE from EF. Figure 17 shows a complete cycle of the graph of (20).

Figure 18 shows the graph of

$$y = x - 2 \sin x \tag{21}$$

in heavy line, obtained by adding ordinates of $y = x$ dotted and $y = -2 \sin x$ in light full line.



$y = \cot^{-1} x$

FIG. 16.

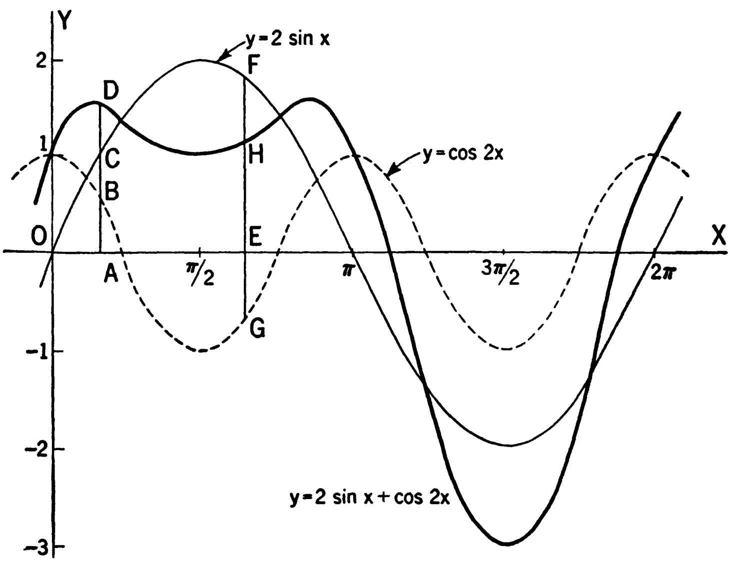


FIG. 17.

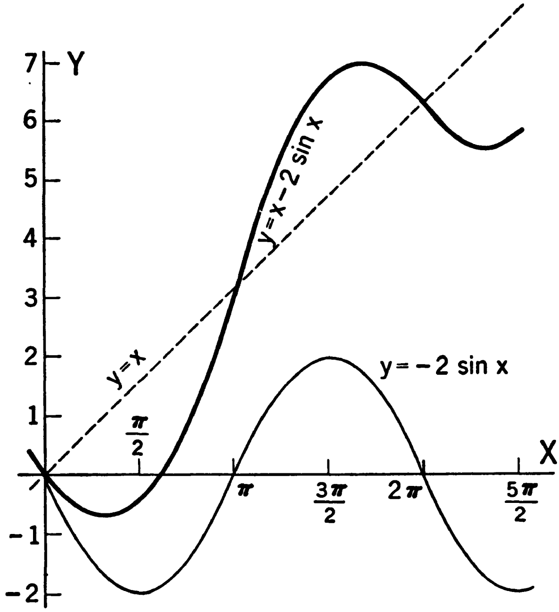


FIG. 18.

Exercises

By using the principle of adding ordinates, sketch the graphs of the equations numbered 1 to 10:

1. $y = \sin x + \cos x.$
2. $y = 2 \cos x + \sin 2x.$
3. $y = x + 2 \cos x.$
4. $y = 1 - \cos x.$
5. $y = 2 \sin x - 1.$
6. $y = x^2 + 2 \cos x.$
7. $y = x - 3 \sin x.$
8. $y = 2 - \sin^{-1} x.$
9. $y = \cos^{-1} 2x - 2.$
10. $y = 3 - \tan^{-1} x.$

For each curve represented by one of the equations numbered 11 to 14, make the indicated translation and sketch:

11. $y = x + 3 - 2 \sin (x - \pi/3)$; $(\pi/3,0)$.
12. $y = \cos (x + \pi/4) - 2 \sin (x + \pi/4)$; $(-\pi/4,0)$.
13. $y = x - 2 \cos (2 - x)$; $(2,0)$.
14. $y = 2x - 3 \sin 2 (x + \pi/6)$; $(-\pi/6,0)$.

Using graphical addition, sketch the graphs of the equations numbered 15 to 19:

15. $\rho = a + a \cos \theta.$
16. $\rho = 2a + a \cos \theta.$

17. $\rho = a + a \sin \theta$.

18. $\rho = a - 2a \sin \theta$.

★**19.** $\rho = 2a \sin \theta + a \sec \theta$.

65. The exponential curves

To plot the curve

$$y = 2^x, \tag{22}$$

form from (22) a table of values such as:

x	-3	-2	-1	0	1	2	3
$y = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

and plot the curve from this in the usual way. The full-line curve in Figure 19 represents the graph. Observe that the x-axis is an asymptote.

The graph of $y = a^x$ has the same general characteristics for any positive value of a except 1. If a is less than unity, the curve is asymptotic to the x-axis on the positive side. The dotted line in Figure 19 represents $y = (\frac{1}{2})^x$. Figure 20 on page 176 shows the graphs of power curves. They all pass through (1,0), any pair $y = m^x$ and $y = (1/m)^x$ are symmetric to each other with respect to the y-axis, and all except $y = 1^x$ have the x-axis as asymptote.

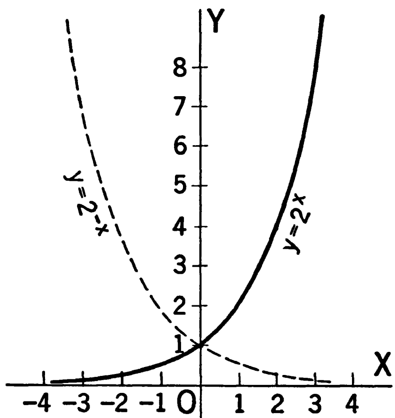


FIG. 19.

A very important number in advanced mathematics and its applications is $e = 2.71828 \cdots$. The graph of $y = e^x$ resembles rather closely that of $y = 3^x$. Two functions, the hyperbolic cosine of x, denoted by $\cosh x$, and the hyperbolic sine of x, denoted by $\sinh x$, are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}. \tag{23}$$

Their graphs may easily be plotted by adding ordinates or by using (23) and the table of values given on page 260.

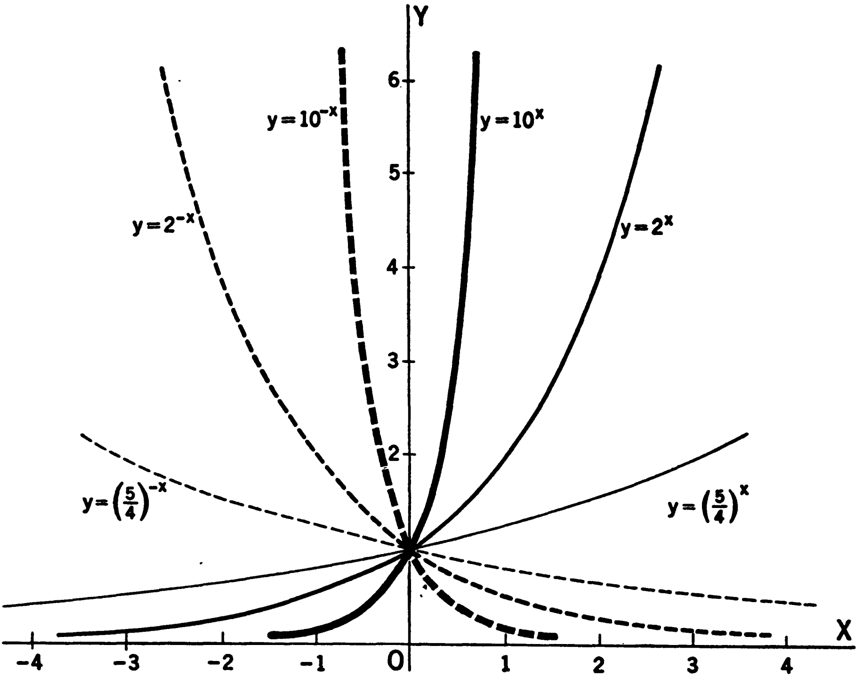


FIG. 20.

66. Logarithmic curves

From the definition of the logarithm of a number,

$$\text{if } y = \log_a x, \quad \text{then } x = a^y. \tag{24}$$

Consider, for example, $y = \log_2 x$. Then, by (24),

$$y = \log_2 x,\ x = 2^y. \tag{25}$$

Assigning values to y and computing x from (25), obtain the table:

y	$-k$	-10	-3	-2	-1	0	1	2	3	10
x	2^{-k}	2^{-10}	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	2^{10}

From this obtain in the usual way the full-line graph shown in Figure 21. From (24) we see that $y = \log_a x$ and $y = a^x$ have the same form, the curve $y = \log_a x$ having the same relation to the x-axis that $y = a^x$ has to the y-axis. In fact, the two curves

are symmetric to each other with respect to the line $y = x$, as indicated in Figure 21 for the case $a = 2$.

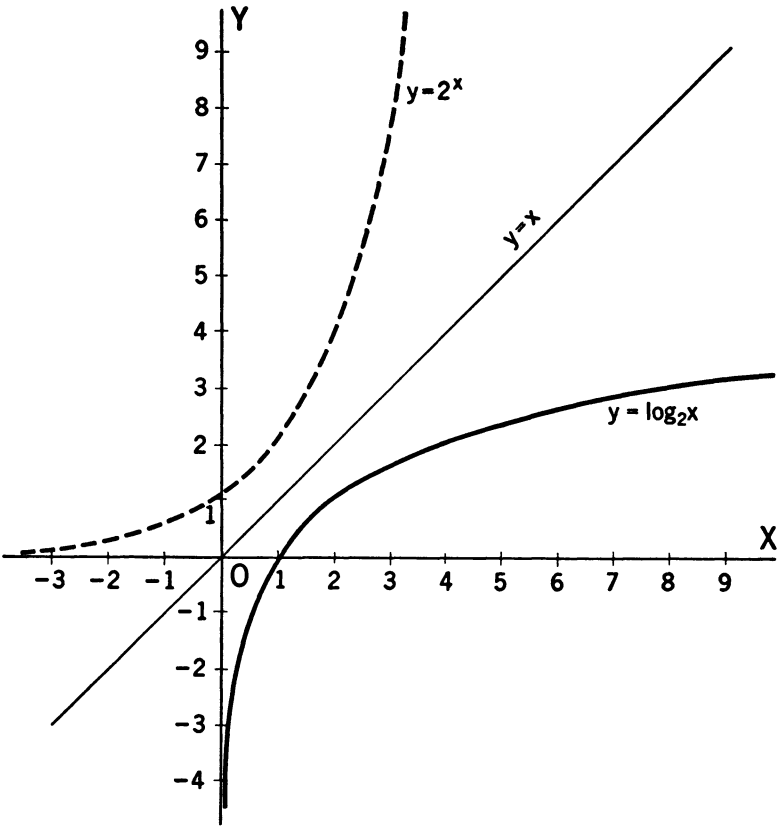


FIG. 21.

Exercises

1. Sketch $y = 3^x$ and $y = (\frac{1}{3})^x$, using the same axes of coordinates.

2. Sketch $y = \log_3 x$. To form a table of values, use $x = 3^y$.

3. Using the table of values of powers of e on page 260, sketch $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$, using the same axes of coordinates.

4. Using the curves of Exercise 3, the definitions 23, and addition of ordinates, plot $y = \cosh x$ and $y = \sinh x$.

5. Sketch on the same coordinate axes $y = (\frac{1}{3})^{x-1}$, $y = 1^{x-1}$, and $y = 3^{x-1}$.

6. Sketch $y - 2 = \log_3 (x - 3)$.

7. Sketch on the same graph $y = \log_{\frac{1}{2}} x$ and $y = \log_2 x$.

Sketch each of the curves numbered 8 to 17:

8. $y = \log_2(-x)$.

9. $y = \log_e(x - 5)$.

10. $y = \log_3(1 + x^2)$.

11. $y = \log_3(x - 3)^2$.

12. $y = \log_{10}(-3x)$.

13. $y = 3e^{-2x}$.

14. $y = 2e^{2x-2}$.

15. $y = 2^{(x-1)^2}$.

16. $y = (1.23)^x$.

17. $y = 5.6\,(1.2)^x$.

18. Sketch the probability curve $y = e^{-x^2}$.

19. The air pressure p pounds per square inch at height h feet is given approximately by

$$p = 15e^{-0.000038h}. \tag{A}$$

Taking $h = 0, 10{,}000, 20{,}000, \cdots, 70{,}000$, compute the corresponding values of p and then plot Equation (A). From your graph read the pressure at the heights 15,000 ft., 25,000 ft., and 65,000 ft.

20. The amount A of radium left after t centuries from an original 100-gram amount is given by

$$A = 100e^{-0.041t}.$$

Graph this equation, and from your graph find the amount remaining after 4 centuries, 11 centuries, and 1360 years.

21. A curve of great importance in the theory of probability is $y = (h/\pi)e^{-hx^2}$. Plot the curve for $h = \frac{1}{2}$, for $h = 1$, and for $h = 2$.

22. By using translation of axes show that the curves $y = \log_{10}x$ and $y = \log_{10}(ax) + c$ are congruent.

★23. By using translation of axes show that the curves $y = a^x$ and $y = ma^x$ are congruent.

67. Curves representing damped vibration

A very important motion of everyday experience is an oscillatory motion which decreases in magnitude and dies away, such as a sound wave or the motion of a weight moving up and down while supported by a spring. It is represented by an equation having the form:

$$y = Ae^{-nx}\sin mx. \tag{26}$$

The **period of oscillation** is $2\pi/m$, and the **damping factor** e^{-nx} decreases with increasing x and approaches zero. In plotting the graph of (26), note that $y = 0$ whenever $\sin mx = 0$, that is, when mx is a multiple of π; and that the graph meets $y = Ae^{-nx}$ or $y = -Ae^{-nx}$ when $\sin mx = \pm 1$, that is, when mx is an odd multiple of $\pi/2$. An approximate graph of the curve may be

obtained by drawing the curves $y = \pm Ae^{-nx}$, marking the points for which $mx = k\pi/2$, k an integer, and then sketching in the curve as indicated in Figure 22. A table of corresponding values of x and y may be used to get the true shape of the curve.

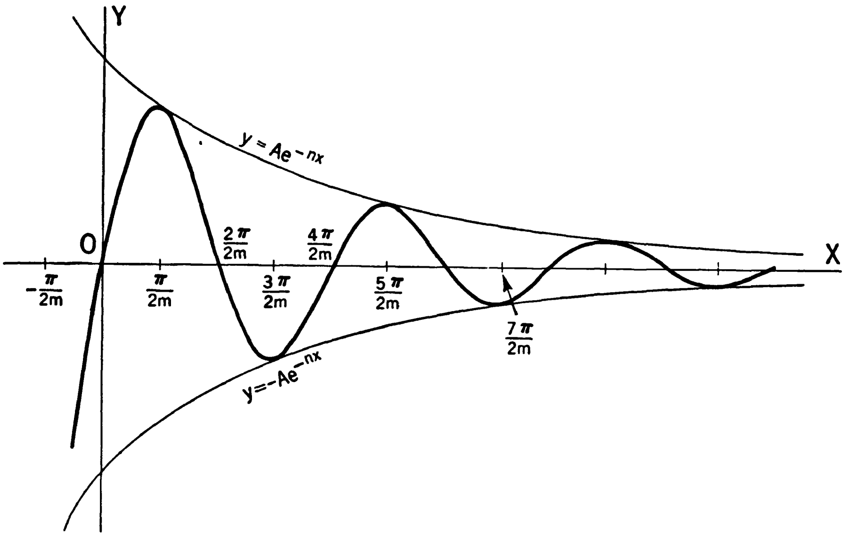


FIG. 22.

Exercises

Sketch each curve represented by the equations numbered 1 to 5 for the indicated range:

1. $y = e^{-x/5} \sin x$; $x = 0$ to 2π.

2. $y = e^{-x/2} \cos x$; $x = -\pi/2$ to 2π.

3. $y = e^{-0.2x} \sin 2x$; $x = -\pi$ to 2π.

4. $y = 5(2)^{-x} \cos (x + \pi/6)$; $x = -\pi/6$ to 2π. First make a translation of axes with new origin $(-\pi/6,0)$.

5. $y = 3e^{-0.3x} \sin (2x - \pi/3)$; $x = -2\pi$ to $\pi/2$. First make a translation of axes with new origin $(\pi/6,0)$.

68. Asymptotes in polar coordinates

Consider the curve represented by

$$\rho = a \cos \theta / \sin \theta. \tag{27}$$

As θ approaches zero, $\sin \theta$ approaches zero and ρ increases

without bound. With no further information, we expect that as θ approaches zero, either

(A) y will become great without bound, as in curve AB, Figure 23,* or

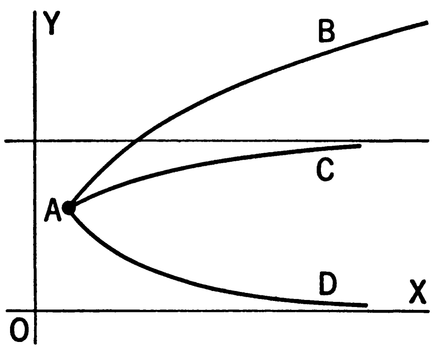


FIG. 23.

(B) y will approach a definite value, as in curve AC, Figure 23. The value in case (B) may be zero, as indicated in curve AD of Figure 23. Each of these possibilities is easily examined by finding

$$y = \rho \sin \theta \qquad (28)$$

and noting its behavior as θ approaches zero.

In the case of Equation (27),

$$y = \rho \sin \theta = (a \cos \theta / \sin \theta)\ (\sin \theta) = a \cos \theta. \qquad (29)$$

Hence, as θ approaches zero, y approaches $a \cos 0 = a$, and case (B) mentioned above applies. Also, as θ approaches π, ρ from (27) increases numerically without bound, but y from (29) approaches $a \cos \pi = -a$. Hence, $y = a$ and $y = -a$ are asymptotes. Observing further that $\rho = 0$ when $\theta = \pi/2$ and $3\pi/2$, and plotting a few points, one can sketch the graph of (27) shown in Figure 24.

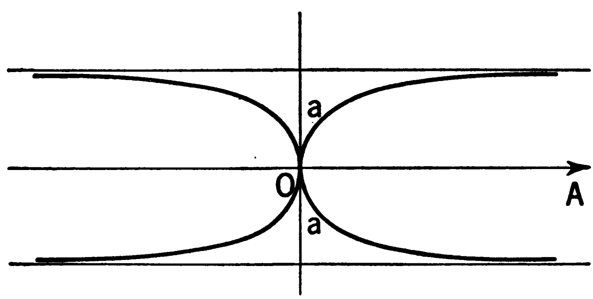


FIG. 24. Graph of $\rho = a \cot \theta$.

Similarly, a curve may be examined for asymptotes perpendicular to the polar axis by finding

$$x = \rho \cos \theta \qquad (30)$$

* This case is illustrated by the parabola $\rho \sin^2 \theta = 2p \cos \theta$ (or $y^2 = 2px$).

for the given curve and noting the value approached by x as θ approaches 90°. By transformation of axes explained in §37, the same idea may be applied to the examination of curves for asymptotes in any direction.

Example 1. Sketch $\rho\theta = a$, where θ is to be expressed in radians. This curve is known as the **hyperbolic spiral.**

Solution. For the given curve, we have

$$y = \rho \sin \theta = a\,(\sin \theta)/\theta. \tag{a}$$

Assuming that $(\sin \theta)/\theta$ approaches 1 as θ approaches zero,* we see from (a) that $y = a$ is an asymptote. Also, from $\rho = a/\theta$ it appears that ρ approaches zero as θ increases without bound. Figure 25 shows the graph.

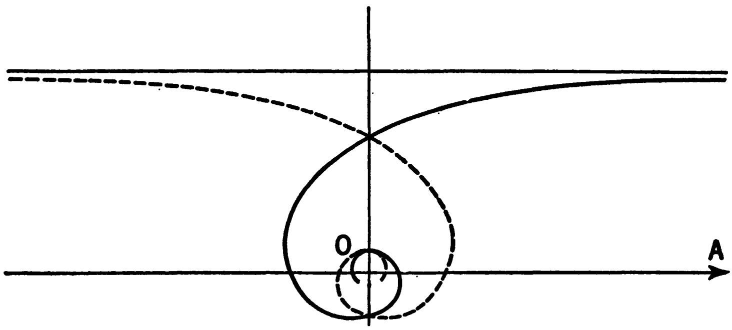


FIG. 25. Hyperbolic Spiral. Graph of $\rho = a/\theta$.

Example 2. Show that $\rho = 2 \tan 2\theta$ has two asymptotes parallel to $\theta = 45°$ and two parallel to $\theta = 135°$, each being distant 1 unit from the pole.

Solution. Observe that $\tan 2\theta$ increases without bound as θ approaches 45°, 225°, 135°, 315°. Hence, replacing θ by $\theta' + 45°$ (see §37), we have

$$\rho' = 2 \tan\,(2\theta' + 90°) = -\,2 \cot 2\theta'. \tag{a}$$

Since

$$\begin{aligned} y' &= \rho' \sin \theta' = -\,\sin \theta'[2 \cos 2\theta'/(2 \sin \theta' \cos \theta')] \\ &= -\,\cos 2\theta'/\cos \theta', \end{aligned}$$

* This statement is proved in books on calculus.

as θ' approaches 0, y' approaches -1; and as θ' approaches 180°, y' approaches $+1$. Hence, $y' = \pm 1$ are asymptotes. Similarly, multiplying (a) through by $\cos \theta'$, we get

$$x' = \rho' \cos \theta' = -\cos 2\theta'/\sin \theta'.$$

Then, as θ' approaches 90°, x' approaches $+1$; and as θ' approaches 270°, x' approaches -1. Hence, $x' = \pm 1$ are asymptotes. The graph is shown in Figure 26.

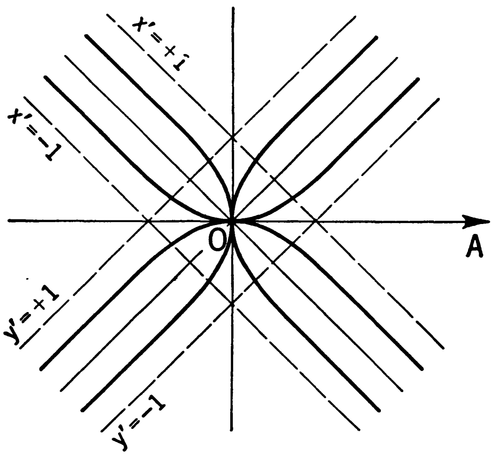


FIG. 26. Graph of $\rho = 2 \tan 2\theta$.

Exercises

1. Show that $\rho = a \tan \theta$ has $\rho \cos \theta$ $(= x) = a$ and $\rho \cos \theta = -a$ as asymptotes.

2. Show that $\rho = a \csc^2 \frac{1}{2}\theta$ does not have an asymptote parallel to $\theta = 0$. Note that $y = \rho \sin \theta = (a/\sin^2 \frac{1}{2}\theta)(2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta)$ increases without bound as θ approaches zero.

3. Show that $\rho \sin 2\theta = a$ has the asymptotes $\rho \sin \theta$ $(= y) = \pm a/2$ and $\rho \cos \theta$ $(= x) = \pm a/2$. Sketch the curve.

Find the equations of all asymptotes and sketch the curves represented by the Equations 4 to 7:

4. $\rho \sin \theta = \cos^2 \theta$.

5. $\rho \cos \theta = \sin^2 \theta$.

★**6.** $\rho^2 \sin^2 \theta = 1 + \cos \theta$.

7. $\rho = \cot \theta$.

8. Find the asymptote of $\rho \sin (\theta - 30°) = \cos^2 \theta$.

9. Find the asymptotes of $\rho \cos 2\theta = a$ and sketch the curve.

Hint. Let $\theta = \theta' + 45°$.

★**10.** Find the asymptotes of $\rho(1 - 2\cos\theta) = a$ and sketch the curve. If $\theta = \theta' + 60°$, then $1 - 2\cos\theta = 1 - 2[\frac{1}{2}\cos\theta' - (\sqrt{3}/2)\sin\theta']$ $= 2\sin^2\frac{1}{2}\theta' + 2\sqrt{3}\sin\frac{1}{2}\theta'\cos\frac{1}{2}\theta' = 2\sin\frac{1}{2}\theta'\,[\sin\frac{1}{2}\theta' + \sqrt{3}\cos\frac{1}{2}\theta']$.

★**11.** Sketch $\rho = 2\cot 2\theta$.

★**12.** Change $x^3 + y^3 = 4a\,xy$ to polar coordinates to obtain $\rho = (4a \sin\theta\cos\theta)/[(\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)]$. Get a loop by inspection. Then get an asymptote parallel to $\theta = 135°$. Sketch the curve. It is called the **Folium of Descartes.**

CHAPTER XI

Curves Representing Empirical Data

69. A typical problem

This chapter deals with laws designed to fit empirical data, that is, data found by measurements of various kinds. Suppose that we wish to obtain a formula giving air pressure as a function of distance from the earth. We could measure the pressures at a great many heights, make a table showing corresponding values of pressure p and height h, plot corresponding points in the usual way, and then draw a smooth average curve near them, some points being above the curve and some below it. Figure 1 shows

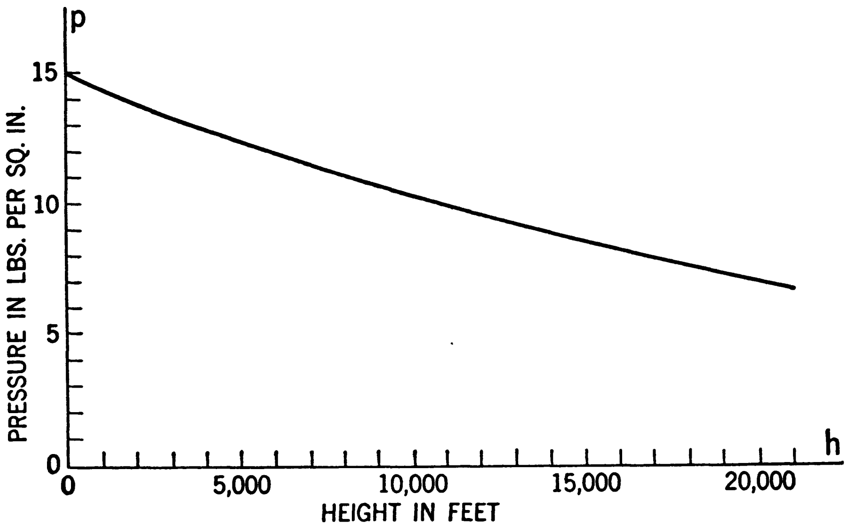


FIG. 1.

a curve obtained in this way. Now, the law connecting pressure and height is very complicated, but, by using the graph, we could estimate the pressures corresponding to various heights.

Thus, from the graph we read $p = 13.5$ lb./in.[2] when $h = 3{,}000$ ft., and $p = 7.7$ lb./in.[2] when $h = 17{,}000$ ft. *The problem is to find a simple equation having a graph closely approximating the curve plotted from the empirical data.* The approximation equation for the data represented by Figure 1 is

$$p = 15e^{-0.000038h}.$$

70. Linear formulas

Suppose that a person driving a tractor along a straight road tabulates distances from his speedometer and corresponding times from his watch and obtains the following table:

Time t (min.)	0	5	10	15	20	25	30
Distance s (miles)	3.00	3.30	3.50	3.70	3.96	4.30	4.55

Figure 2 represents the value pairs as points, and a straight line near the points indicates a relation between distance and time.

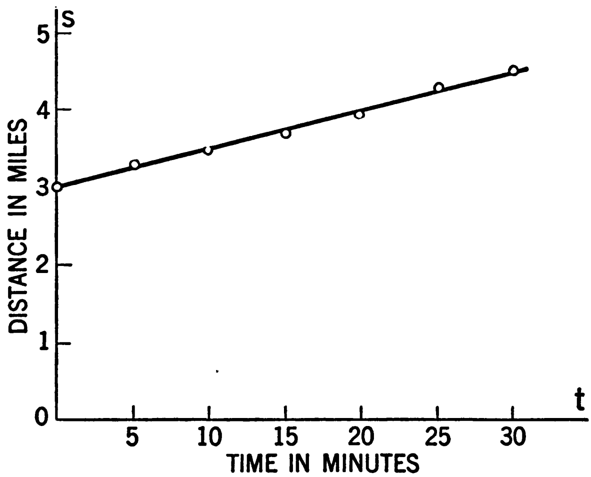


FIG. 2.

We seek the equation of a straight line which runs near the points as indicated.

Let

$$s = at + b \tag{1}$$

be the equation of the desired line. Here, as in the remaining part of the chapter, we use the term **residual** to mean *the directed vertical distance from the plotted point to the approximating curve.* Then, the residual r_i of the point (s_i,t_i) is

$$r_i = s_{\text{point}} - s_{\text{line}} = s_{\text{point}} - at_i - b.$$

Applying this formula to the values in the table, we get

$$\begin{aligned} r_1 &= 3.00 - 0a - b, & r_5 &= 3.96 - 20a - b, \\ r_2 &= 3.30 - 5a - b, & r_6 &= 4.30 - 25a - b, \\ r_3 &= 3.50 - 10a - b, & r_7 &= 4.55 - 30a - b. \\ r_4 &= 3.70 - 15a - b, \end{aligned} \tag{2}$$

Some of the residuals should be negative and some positive, and it seems reasonable that the negative residuals should balance the positive ones. Also, we need two equations in a and b. To get them, we take

$$r_1 + r_2 + r_3 + r_4 = 0, \ r_5 + r_6 + r_7 = 0, \tag{3}$$

or, replacing the r's by their values from (2) and simplifying:

$$13.50 - 30a - 4b = 0, \ 12.81 - 75a - 3b = 0. \tag{4}$$

The solution of these equations for a and b is

$$a = 0.0511, b = 2.99. \tag{5}$$

Substitution of these values in (1) gives

$$\mathbf{s = 0.0511t + 2.99.} \tag{6}$$

This method of dividing the residuals into two groups and setting the sum of each group equal to zero is called the **method of averages.** The number of residuals in each group should not differ by more than 1. Different groupings give different formulas; but, in general, for data fitting a linear formula approximately, they do not differ materially. A method giving a formula considered *the best* is given in §72.

Example. The horsepower H required to drive a steamship at speed v knots is given approximately by a formula of the type

$$H = a + bv^3. \tag{a}$$

Find, by the method of averages, a formula for the data of the table:

v	5	7	9	11	12
H	290	560	1040	1810	2300

(b)

Solution. The method of averages requires a linear formula. Hence, let $V = v^3$. This gives the table of values:

$V(=v^3)$	125	343	729	1331	1728
H	290	560	1040	1810	2300

(c)

and we use the formula $H = a + bV$. Equating to zero the sum of the first three residuals and the sum of the last two, we get

$$\left.\begin{aligned}(290 - a - 125b) + (560 - a - 343b)&\\ + (1040 - a - 729b) &= 0\\ (1810 - a - 1331b) + (2300 - a - 1728b) &= 0.\end{aligned}\right\} \quad \text{(d)}$$

The solution of these equations is $a = 127$, $b = 1.26$. Substituting these values in (a), we get

$$\mathbf{H = 127 + 1.26v^3}. \qquad \text{(e)}$$

71. Accuracy

A thorough discussion of the accuracy to be obtained by using these formulas would be rather complicated and will not be given here. In this book we shall understand that values of the independent variable, given in the first line of data, are to be considered exact. *The formulas obtained should be such as to give results of the same accuracy as that of the figures in the second line of data.* If the figures in the second line of data are accurate to three figures, as in the first illustration of §**70**, then the formula should show numbers accurate to three figures. In determining the number of figures accurate in data, a first figure 1 is often not considered as a figure representing accuracy; thus, 4.0, 2.7, 0.0027, 11.5, and 0.00115 are accurate to two figures.

In computing a result, it is well to keep one more figure than required by the formula and then round off the final results to

the desired number of figures. This was done in the example of §70.

Exercises

1. Find an approximate formula having the form $y = ax + b$ for the data of the following table:

x	2	4	6	8	10	12	14
y	4.0	6.0	8.4	9.0	10.5	14.5	15.0

2. Find a linear formula for approximating the specific gravity y of dilute sulphuric acid at different concentrations x per cent, from the table:

x	5	15	25	35
y	1.033	1.101	1.178	1.257

3. Find an empirical formula of the type $R = aT + b$ for the resistance R ohms of a copper wire at temperature $T°$ Cent. from the table of values:

T	20	25	30	40	45	50
R	76	78	80	82	84	85

4. Find an empirical formula of the type

$$w = b + a/I$$

for the data given in the table:

I	12.5	13.1	14.1	16.3
w	36.8	26.3	15.8	8.4

Hint. Make a new line for Z in the table, where $Z = 1/I$, and then find a linear formula of the type $w = b + aZ$.

5. The force F required to lift a block of weight W by means of a pulley is approximately of the form $F = a + bW$. Find an empirical formula by using the data:

W	10	20	30	50	70	90
F	3.25	4.87	6.25	9.00	12.25	15.00

6. If H is the total heat in a pound of saturated steam, and T the temperature in degrees centigrade, $H = aT + b$, approximately. Find H in terms of T for the data:

T	65	85	100	120
H	626	632	637	643

7. Within certain limits, the length l of a cable is related to the weight W which it supports by a linear formula $l = aW + b$. Find the formula from the data of the following table:

W lb.	1,000	3,000	6,000	7,000	9,000	11,000
$(l - 1{,}200)$ in.	0.04	0.13	0.25	0.30	0.35	0.46

8. Find an empirical formula of the type $y = ax^2 + b$ to fit the following data:

x	1	2	3	4	5
y	-1.00	5.3	14.5	30	46

9. The lengths L of a certain spring (Figure 3) supporting various loads W were measured and the results tabulated as follows:

W lb.	60	100	140	180	220	260
L ft.	10.25	10.55	10.73	10.85	11.07	11.33

Find the empirical formula for L in terms of W.

10. When a parachute falls in air, the pressure p on the parachute in pounds per square inch and the velocity v of the parachute in feet per second are connected approximately by the law

$$p = a + bv^2.$$

Find the formula by using the data

v	8	11.5	16.4	22.6	32.8
p	0.20	0.40	0.80	1.60	3.2

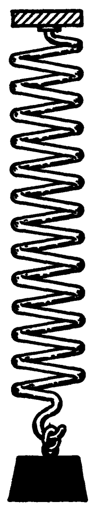


FIG. 3.

72. The method of least squares for linear relations

A linear relation based on the result of setting sums of residuals equal to zero is not definite, since different groupings give different results. A definite line, and generally a better one, is determined by specifying that the *sum of the squares of the residuals of the measured points be least.* The method of finding this line is called the **method of least squares.** The method will be explained in this article and proved in the next one. In stating the method, it will be convenient to use the notation

$$\Sigma x_i = x_1 + x_2 + \cdots + x_n, \tag{7}$$

$$\Sigma x_i^2 = x_1^2 + x_2^2 + \cdots + x_n^2, \tag{8}$$

$$\Sigma x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n. \tag{9}$$

Assume that sets of value pairs satisfying, approximately, a linear equation

$$y = ax + b \tag{10}$$

are given by the table

x	x_1	x_2	x_3	$\cdots x_n$
y	y_1	y_2	y_3	$\cdots y_n$

(11)

Substitute each pair in (10) to obtain the approximate equations

$$\begin{aligned} y_1 &= ax_1 + b, \\ y_2 &= ax_2 + b, \\ &\cdots\cdots \\ y_n &= ax_n + b. \end{aligned} \tag{12}$$

In practice these equations, called *observational* equations, are written for convenience of statement and form. They are only approximately true. The normal equations used in the theory are obtained from them, but there is no stipulation that each observational equation be exactly true.

Add the equations (12) member by member to obtain

$$\Sigma y_i = a\,\Sigma x_i + nb. \tag{13}$$

Multiply the first equation of (12) by x_1, the second by x_2, $\cdots$, the last one by x_n, and add the equations member by member to obtain

$$\Sigma x_i y_i = a\,\Sigma x_i^2 + b\,\Sigma x_i. \tag{14}$$

The required equation is obtained by solving Equations (13) and (14) for a and b and substituting the results in (10).

Equations (13) and (14) are called the **normal equations.** The following solution illustrates the plan to be used in finding an approximating linear formula by the method of least squares.

Example. Using the method of least squares, find a linear relation satisfied approximately by the following data:

x	1	2	3	4	5
y	4.00	6.50	8.50	11.80	14.00

Solution. Writing five equations corresponding to (12), and then multiplying each equation by the coefficient of a in that equation, obtain

$$\begin{aligned} 4.0 &= 1a + b & 4.0 &= a + b \\ 6.5 &= 2a + b & 13.0 &= 4a + 2b \\ 8.5 &= 3a + b & 25.5 &= 9a + 3b \\ 11.8 &= 4a + b & 47.2 &= 16a + 4b \\ 14.0 &= 5a + b & 70.0 &= 25a + 5b \end{aligned} \tag{a}$$

Add the equations in each column member by member to obtain the normal equations

$$44.8 = 15a + 5b, \quad 159.7 = 55a + 15b. \tag{b}$$

Solve these equations for a and b, and substitute the results in (10) to obtain the required equation

$$\boldsymbol{y = 2.53x + 1.37.} \tag{c}$$

Observe that *the method of finding a formula of the type* $y = ax + b$ *for given observed value pairs* x_i *and* y_i *consists in substituting the value pairs in* $y = ax + b$ *to form observational equations, adding the observational equations to find one normal equation, multiplying each observational equation by the coefficient of* a*, adding the results to get the second normal equation, solving the normal equations for* a *and* b*, and substituting the results in* $y = ax + b$.

73. Derivation of the normal equations

In the following derivation, the abscissa of the vertex of a parabola

$$Y = lX^2 + mX + n \tag{15}$$

will be needed. From (15), we have

$$Y - n + \frac{m^2}{4l} = l\left(X^2 + \frac{m}{l}X + \frac{m^2}{4l^2}\right) = l\left(X + \frac{m}{2l}\right)^2. \tag{16}$$

Since the parabola $(x - h)^2 = 2p(y - k)$ has (h,k) as its vertex, we see that X_v of the vertex of (16) is given by

$$X_v = -m/(2l). \tag{17}$$

The value pairs, or points, given by Table (11), §**72,** satisfying approximately an equation of the form (10), §**72,** have residuals $r_1, r_2, \cdots, r_n$, defined by

$$\begin{aligned} r_1 &= y_1 - ax_1 - b, \\ r_2 &= y_2 - ax_2 - b, \\ &\cdots\cdots\cdots\cdots \\ r_n &= y_n - ax_n - b. \end{aligned} \tag{18}$$

The sum, Σr_i^2, of the squares of these residuals is to be as small as possible. From (18),

$$\Sigma r_i^2 = a^2\Sigma x_i^2 - 2a(\Sigma x_i y_i - b\Sigma x_i) + \Sigma(y_i - b)^2. \tag{19}$$

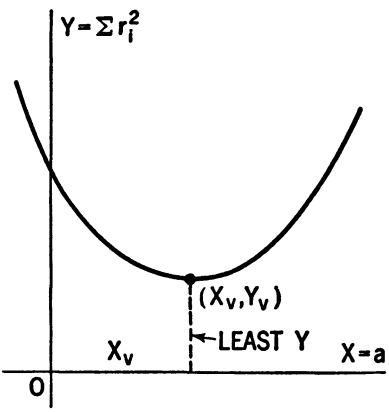


FIG. 4.

Letting $\Sigma r_i^2 = Y$ and $a = X$ in (19), we see that its graph has the form of a parabola with no negative ordinates (see Figure 4). $Y = \Sigma r_i^2$ is least at the vertex of the parabola. Hence, for Σr_i^2 least we must have, in accordance with (17),

$$X = a = \frac{2[\Sigma x_i y_i - b\Sigma x_i]}{2\Sigma x_i^2} \tag{20}$$

or

$$\Sigma x_i y_i = a\Sigma x_i^2 + b\Sigma x_i, \tag{21}$$

and this is Equation (14). Similarly, by arranging the expression for Σr_i^2 in (19) in powers of b instead of a, and repeating the argument used to derive (21), we obtain Equation (13).

For the example of §72, the respective residuals are: $r_1 = 4 - 2.53 - 1.37 = 0.10$, $r_2 = 6.50 - 2.53(2) - 1.37 = 0.10$, $r_3 = -0.46$, $r_4 = 0.31$, $r_5 = -0.02$. For these, we have

$$\Sigma r_i^2 = 0.3281.$$

Using the method of averages on the example of §72, by taking as groups the first three observational equations and the last two, we get

$$y = 2.63x + 1.08.$$

Computing the residuals for this, we get $r_1 = 0.29$, $r_2 = 0.16$, $r_3 = -0.47$, $r_4 = 0.20$, and $r_5 = -0.23$. For these,

$$\Sigma r_i^2 = 0.4235.$$

Comparing these two values of Σr_i^2, we see that the Σr_i^2 for the least-square answer is considerably smaller than Σr_i^2 for the answer obtained by the method of averages; accordingly, we call the least-square answer better.

Exercises *

1. Find by the method of least squares an empirical equation of the form $y = ax + b$ for the data

x	1	2	3	4	5	6
y	4.8	7.3	9.4	10.6	13.0	15.6

2. The lengths l inches of a spring under various loads w are given by the table

w	10	20	30	40	50	60
l	10.4	11.2	11.7	12.0	12.4	13.1

Assume a formula $l = aw + b$ and use the method of least squares to get the empirical formula.

* The answers to all problems in this and the next two articles will be computed by the method of least squares.

★**3.** The speed v of a steamship varies with the horsepower H according to the law

$$H = av^3 + b.$$

Given the following data:

v knots	5	7	9	11	12
H	290	560	1040	1810	2300

make a new row from the relation $V = v^3$ and then use least squares to find a and b in $H = av^3 + b = aV + b$.

4. The lengths L of a certain coiled steel spring when supporting various weights w were tabulated as follows:

w	0	10	20	30	40	45
L	10.3	12.8	15.7	18.6	21.2	23.0

Find a linear formula to fit the data.

5. The pressure p in inches of mercury of a gas remaining at constant volume varied with the temperature T in degrees centigrade as follows:

T	20	30	40	50	60	70	80
p	27.1	28.9	30.6	32.3	33.7	35.6	37.2

Find a linear formula for p in terms of T.

In Exercises 6 to 13 apply the method of least squares to the indicated data:

6. Exercise 1, §71.

7. Exercise 2, §71.

8. Exercise 3, §71.

9. Exercise 4, §71.

10. Exercise 5, §71.

11. Exercise 6, §71.

12. Exercise 7, §71.

13. Exercise 8, §71.

14. Solve the example of §70, using the method of least squares.

74. Formulas of the type $y = ax^n$

Many physical relations are approximated by an equation having the form

$$y = ax^n. \tag{22}$$

In this case, a and n are to be found. Equate the logarithms to the base 10 of the two members, obtaining

$$\log_{10} y = n \log_{10} x + \log_{10} a, \tag{23}$$

make the substitution

$$Y = \log_{10} y, \; X = \log_{10} x, \; A = \log_{10} a \tag{24}$$

in (23) to obtain

$$Y = nX + A, \tag{25}$$

and then use the method of averages, §70, or the method of least squares, §72, to obtain n and $\log_{10} a$ from the linear form (25).

When data satisfying approximately an equation of the type (22) is plotted on logarithmic paper (see Figure 5), the points lie nearly in a straight line. This paper has vertical and horizontal rulings spaced at intervals of log 2, log 3, log 4, · · · , log 10, log 20, log 30, · · · , log 100, and so on. Since the graph on logarithmic paper is made from the original data, it provides a short way of determining whether the data are of a type approximated by an equation having the form (22). To get n and a approximately from such a graph, take for n the actual slope of the line and for a the y-intercept read from the logarithmic scale on the y-axis. An example will illustrate the methods just discussed.

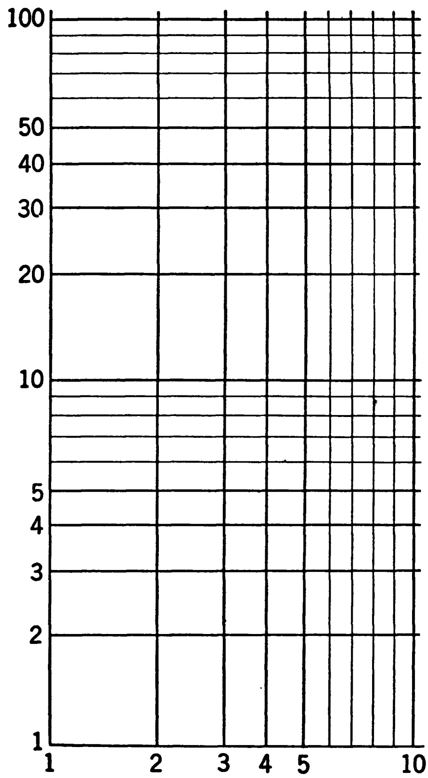


FIG. 5.

Example. When superheated steam expands without loss of energy, it obeys the law

$$p = av^n, \tag{a}$$

where p is the pressure per square unit on the steam and v is its volume. Find the law of expansion from the empirical data for steam in the following table:

v (cu. ft.)	50	40	30	20	15	10	5
p (lb./in.^{2})	4.0	5.5	8.3	15	22	39	103

Solution. Figure 6 shows the data plotted on logarithmic paper. Note that the slope of the line is about -1.4 and the p-intercept is about 1,000. Hence, the law is

$$p = 1{,}000v^{-1.4},$$

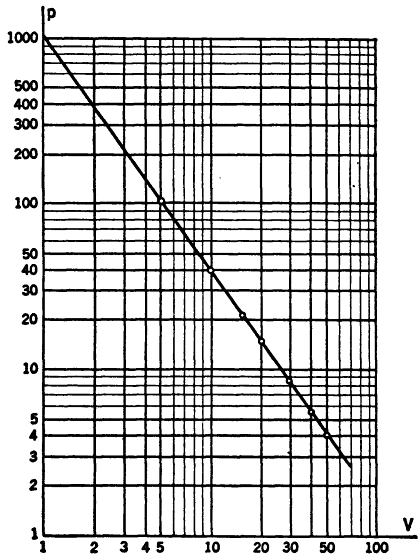


FIG. 6.

approximately. However, let us apply the method of least squares. First construct new data by taking the logarithms to base 10 of the given data. This gives

$V = \log_{10} v$	1.699	1.602	1.477	1.301	1.176	1.000	0.699 *
$P = \log_{10} p$	0.602	0.740	0.919	1.176	1.342	1.591	2.013

* Mantissas of logarithms to three decimal places were used, but final results will be rounded to two figures to correspond with the accuracy of the given data.

Forming observational equations from $P = nV + A$, and forming a new set from them by multiplying each by the coefficient of n in it, we get

$$
\begin{aligned}
0.602 &= 1.699n + A & \qquad 1.023 &= 2.887n + 1.699A \\
0.740 &= 1.602n + A & 1.185 &= 2.566n + 1.602A \\
0.919 &= 1.477n + A & 1.357 &= 2.182n + 1.477A \\
1.176 &= 1.301n + A & 1.530 &= 1.693n + 1.301A \\
1.342 &= 1.176n + A & 1.578 &= 1.383n + 1.176A \\
1.591 &= 1.000n + A & 1.591 &= 1.000n + 1.000A \\
2.013 &= 0.699n + A & 1.407 &= 0.489n + 0.699A
\end{aligned}
$$

Adding these equations, we get the normal equations

$$8.383 = 8.954n + 7A, \quad 9.671 = 12.200n + 8.954A.$$

The solution of these equations is $A = 3.0$, $n = -1.41$. Hence, from (24), $\log_{10} a = 3.0$, $a = 1{,}000$. Substituting $a = 1{,}000$, $n = -1.41$ in (a), we have

$$p = 1{,}000v^{-1.41}.$$

Exercises

For the sets of data numbered 1 to 3, assume a law expressed by $y = ax^n$ and, using the least-squares method or the method of averages, find the formula which fits the data:

1.

x	1	10	100	1000
y	1.9	6.3	20	64

2.

x	1	10	100	1000
y	300	9.5	0.32	0.011

3.

x	54	26	14	7.0
y	6.9	15	29	60

4. The resistance R in pounds to an automobile at speed V miles per hour obeys approximately the law $R = AV^n$. Find the law for the following data:

V	10	20	40	60
R	6.40	25.2	108	232

5. Measurements of volume and pressure of steam were made as indicated by the table:

v (ft.^{3})	10	5	2.5	2	1
p (lb./in.^{2})	5	12	30	40	100

Find a formula of the type $p = av^n$ fitted to the data.

6. The force F of attraction and the distance S between two magnetic poles is measured as indicated in the following table:

S (cm.)	0.5	1.0	1.5	2.0	2.5
F (dyne)	15.5	3.3	1.8	0.83	0.63

Find a formula of the type $F = aS^n$ expressing the relation.

7. The quantity Q tons of coal consumed by a certain locomotive varied with the velocity v as indicated in the following table:

v (mi./hr.)	20	30	40	50	60
Q (tons)	0.9	2.0	3.6	5.6	8.1

Find a formula of the type $Q = av^n$ expressing the relation.

8. The adjoining table gives for each major planet its mean distance r from the sun in millions of miles and its period T years. Find the law expressing T in terms of r.

r	36	67	93	142	483	886	1782	2792
T	0.24	0.62	1.00	1.88	11.9	29	84	165

9. The water surface in a reservoir stands h ft. above an orifice 1 inch in diameter. Find the discharge D in terms of h from the following table of values:

h(ft.)	1	3	5	7
D(ft.^{3}/min.)	2.0	3.5	4.5	5.3

75. Formulas of the exponential type

If the formula required to represent data has the form

$$y = a \cdot 10^{bx} \tag{26}$$

approximately, the plan of §**74** is used. Equate the logarithms of the two members of (26) to obtain

$$\log_{10} y = bx + \log_{10} a. \tag{27}$$

Replace $\log_{10} y$ by Y in (27), obtaining

$$Y = bx + \log_{10} a = bx + A, \tag{28}$$

and then use the method of §**70** or of §**72** on the data $Y(=\log_{10} y)$ and x to find b and $A(=\log_{10} a)$ from the linear form (28).

Figure 7 shows a piece of semi-logarithmic paper, that is, paper with vertical lines spaced at equal intervals along the horizontal axis, and horizontal lines spaced at intervals of log 2, log 3, log 4, · · · , log 10, log 20, · · · , along the vertical axis. The graph of data obeying the law (26) will be a straight line on semi-logarithmic paper if the original data for y are laid off on the logarithmic scale and the data for x on the uniform scale.

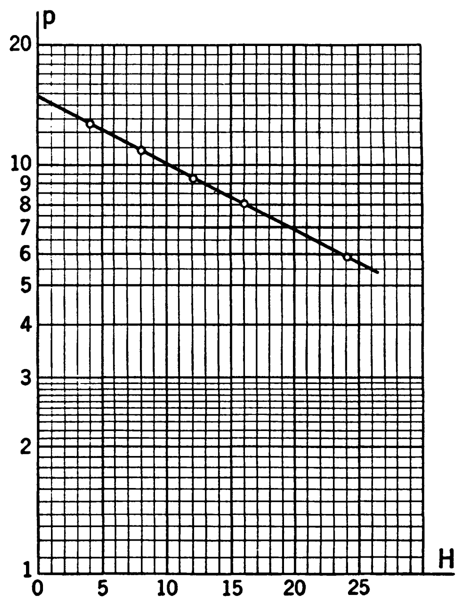


FIG. 7.

Example. The air pressure p at various heights H in thousands of feet is given by the table

H (h/1,000 ft.)	4	8	12	16	24
p (lb./in.^{2})	12.6	10.8	9.3	8.0	5.9

Find p as a function of H in the form $p = a\,10^{bH}$.

Solution. Figure 7 shows the graph as a straight line on semilogarithmic paper. Letting $P = \log_{10} p$, we get the data

H	4	8	12	16	24
P	1.100	1.033	0.968	0.903	0.771

to go with the formula $P = \log a + bH = A + bH$. Forming the normal equations in the usual way, we get

$$\begin{array}{rcl} 1.100 &=& A + 4b \\ 1.033 &=& A + 8b \\ 0.968 &=& A + 12b \\ 0.903 &=& A + 16b \\ \underline{0.771} &=& \underline{A + 24b} \\ 4.775 &=& 5A + 64b \end{array} \qquad \begin{array}{rcl} 4.40 &=& 4A + 16b \\ 8.26 &=& 8A + 64b \\ 11.62 &=& 12A + 144b \\ 14.45 &=& 16A + 256b \\ \underline{18.50} &=& \underline{24A + 576b} \\ 57.23 &=& 64A + 1056b \end{array}$$

The solution of these equations is $A = 1.168$, $b = -0.0164$; therefore, $\log_{10} a = 1.165$, $a = 14.6$, and the required formula is $p = \mathbf{14.6 \cdot 10^{-0.0164H}}$.

Exercises

Using the data of the tables numbered 1 to 3, find formulas for y in terms of x having the form $y = a \cdot 10^{bx}$.

1.

x	1	2	3	4	5
y	35	19.8	10.9	6.3	3.5

2.

x	0	3	6	9	12
y	75	54	39	28	20

3.

x	0	1	2	3	5
y	880	660	500	380	210

4. A chemical reaction gave rise to the data of the table

t	2	8	14	27
A	95	81	69	49

Find a corresponding formula of the type $A = a \cdot 10^{bt}$.

5. The table represents the number N of bacteria found in a culture after t hours:

t	1	3	5	7
N	73	112	163	241

Find N in terms of t.

6. The temperature T of a cooling body at time t was recorded in the following table:

t (min.)	0	3	6	9	12	15
T (degrees C)	100	62	39	24	14.8	9.3

Find a formula connecting t and T.

7. The speed v of a boat at time t was recorded as follows:

t (sec.)	0	3	6	9	12	15
v (ft./sec.)	44	22	11.0	5.6	2.9	1.49

Find v in terms of t.

76. Formulas of the polynomial type

When no one of the types of formulas already considered gives a sufficiently accurate result, an equation having the form

$$y = a + bx + cx^2 + \cdots + hx^n \tag{29}$$

may be tried. A sketch of the data will suggest the degree n to be used. If $n = 2$, Equation (29) becomes

$$y = a + bx + cx^2. \tag{30}$$

To find a, b, and c for a set of data fitting this formula, first form observational equations by substituting the value pairs in (30). *By the method of averages,* divide the observational equations into three groups, add the equations of each group, and solve the three resulting equations simultaneously for a, b, and c. *By the method of least squares,* add the observational equations, multiply each equation by the coefficient of b and add, and multiply each equation by the coefficient of c and add; solve the resulting normal equations for a, b, and c. A corresponding extension applies for data fitting a polynomial of any degree.

Example. Find a formula of the type (30) for the data

x	0	2	4	6	8	10
y	5.0	7.3	11.4	17.0	24	33

Solution. Substitute the values from the table in (30) to obtain

$$\begin{cases} 5.0 = a + 0b + 0c \\ 7.3 = a + 2b + 4c \end{cases}$$
$$\begin{cases} 11.4 = a + 4b + 16c \\ 17.0 = a + 6b + 36c \end{cases} \tag{a}$$
$$\begin{cases} 24 = a + 8b + 64c \\ 33 = a + 10b + 100c \end{cases}$$

Grouping the equations as indicated by the braces, and adding the equations in each group, obtain

$$12.3 = 2a + 2b + 4c, \qquad 28.4 = 2a + 10b + 52c,$$
$$57 = 2a + 18b + 164c. \tag{b}$$

The solution of Equations (b) is

$$a = 4.9,\ b = 0.84,\ c = 0.195.$$

Hence, the required formula is $y = \mathbf{4.9 + 0.84x + 0.195x^2}$.

Exercises

1. Find a formula of type (30) to fit the data:

x	0	1	2	3	4	5
y	2.0	2.6	3.3	4.6	6.3	8.6

2. Find a formula of type (30) to fit the data:

x	0	2	4	6	8	10
y	0.49	−1.13	−1.88	−1.92	−1.12	0.51

★3. Find a formula of type (30) to fit the data:

x	87.5	84.0	77.8	63.7	46.7	36.9
y	292	283	270	235	197.0	181.0

77. Miscellaneous problems on empirical formulas

If the form of an empirical equation to fit given data is unknown, the following procedure may be used:

1. Plot the points representing the data on ordinary coordinate paper. If the points are practically in line, use a linear formula. (§§70, 72)

2. Plot the data on logarithmic paper. If the resulting points are nearly in line, use a formula of the type $y = ax^n$. (§74)

3. Plot the data on semi-logarithmic paper. If the resulting points are nearly in line, use a formula of the type $y = a \cdot 10^{bx}$. (§75)

4. If the directions numbered 1, 2, and 3 lead to no solution, then try a formula of type (30), §76.*

---

* There are methods of telling when a polynomial type of formula is fitted to data. They are, however, rather complicated and have not been given.

Exercises

Find an empirical formula to fit each of the following sets of data:

1.

x	2	4	6	8	10
y	1.30	1.77	2.4	2.7	3.3

2.

x	1	2	3	4	5	6
y	3.5	5.1	6.4	7.5	8.5	9.1

3.

x	0	2	4	6	8	10
y	9.8	6.8	4.5	3.0	2.1	1.4

4.

x	0	1	2	3	4	5
y	3.0	8.3	18	31	45	67

5.

t	1	3	5	7	8
N	1491	3390	7650	17,320	26,100

These data relate to the growth of bacteria.

6.

t	0	2	4	6	8	10
T	64.9°	55.0°	47.2°	41.9°	37.6°	33.4°

These data relate to a body cooling in air.

7.

T	97.5	132.5	162.5	185.0	195.0
p	1	3	6.8	11.5	14.3

These data relate to pressure and temperature of saturated steam.

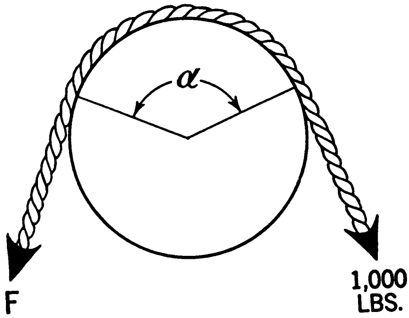


FIG. 8.

8.

α (radians)	1	2	3	4	5	6
F (lbs.)	607	368	223	135.6	82.1	49.8

This table shows the force F required to hold a rope from slipping due to a 1,000-lb. pull when wrapped at various angles α about a certain tree (see Figure 8).

CHAPTER XII

Graphs and Loci in Space

78. Rectangular coordinates in space

Take three mutually perpendicular lines, or axes (see Figure 1), in space, all passing through point O. Assign to each a positive direction and associate numbers to its points, as was done in §3,

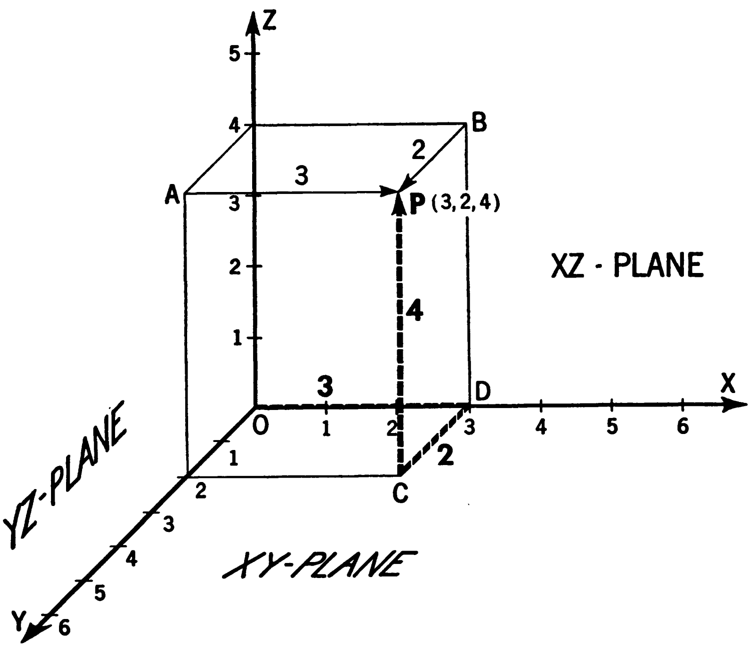


FIG. 1.

Chapter I, zero being assigned to point O, in each case. Then one of these axes (OX, in Figure 1) is called the **x-axis**, a second one is called the **y-axis**, and the third the **z-axis**. The point O is called the **origin**. The planes containing the x-axis and the y-axis, the x-axis and the z-axis, the y-axis and the z-axis, are called the **xy-plane**, the **xz-plane**, and the **yz-plane**, respectively, and are referred to collectively as **the coordinate planes**.

To each number triplet (x,y,z) is associated the point having directed distances of x,y, and z from the yz-plane, the xz-plane, and the xy-plane, respectively. The first number is called the **x-coordinate,** the second the **y-coordinate,** and the third the **z-coordinate,** and the three numbers are called the coordinates of the point represented. These coordinates for the point $P(3,2,4)$ are represented in Figure 1 by the vectors $\overrightarrow{AP}$, $\overrightarrow{BP}$, and $\overrightarrow{CP}$, respectively. To every number triplet is thus associated a definite point in space, and to every point in space is associated a unique number triplet. To plot the point (3,2,4), we do not draw a box as in Figure 1, but we draw the broken-line path $ODCP$; a like remark applies to any other point.

The eight parts into which space is divided by the coordinate planes are called **octants.** The octant containing the points having all of their coordinates positive is called the **first octant.**

The xy-plane is generally considered as being horizontal, the x-axis is considered as directed rightward, and the z-axis as directed upward. Thus, we speak of points having a positive x-coordinate as being to the right of the yz-plane, and points with positive z-coordinate as being above the xy-plane. The student may find it helpful at first to think of a corner of a room as origin, the floor as the xy-plane, and the two vertical walls through the corner as the xz-plane and the yz-plane.

Figure 2 on page 208 shows several points plotted.

Exercises

1. Draw a system of coordinate axes making the angle (representing 90°) between the x-axis and the y-axis about 135°. Plot the points (1,2,2), (1,−2,2), (1,2,−2), (−1,3,−2), and (−2,−1,−1).

2. What are the coordinates of: (a) the origin? (b) the point on the z-axis 3 units above the xy-plane? (c) the point on the x-axis a units right of the yz-plane?

3. Using a to represent an arbitrary number, express the coordinates of: (a) the points on the x-axis; (b) the points on the y-axis; (c) the points on the z-axis; (d) the points in the xy-plane with abscissa 2.

4. Where do all points lie having: (a) the x-coordinate positive? (b) the z-coordinate negative? (c) the y-coordinate positive?

5. Where do all points lie for which: (a) $x = 0$? (b) $y = 0$? (c) $z = 0$?

6. Where do all points lie having: (a) x-coordinate 2? (b) y-coordinate 3? (c) z-coordinate −3? (d) x-coordinate −3?

7. If k represents an arbitrary number, what is the locus of all points representable by: (a) $(0,0,k)$? (b) $(0,1,k)$? (c) $(1,2,k)$? (d) $(0,k,0)$? (e) $(k,1,2)$?

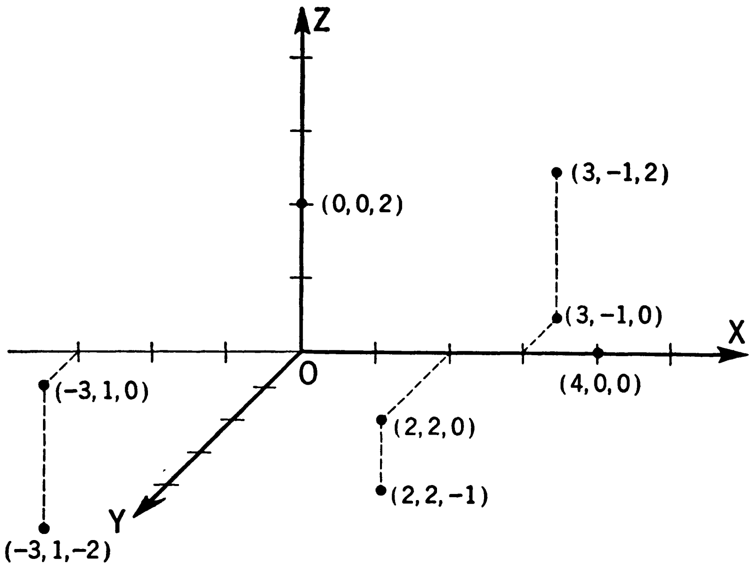


FIG. 2.

79. Equations of surfaces

Just as equations in plane analytic geometry generally represent curves, equations in three coordinates generally represent surfaces. *A surface is represented by an equation if every point on it has coordinates which satisfy the equation and if every point whose coordinates satisfy the equation lies on the surface.*

Evidently $z = 2$ represents a plane parallel to the xy-plane and 2 units above it, since all points in this plane have $z = 2$ and no others have it. Similarly, $x = -10$ represents a plane 10 units to the left of the yz-plane, and if k is a positive number, $y = k$ is a plane parallel to the xz-plane and k units from it on the positive side.

A surface consisting of all lines perpendicular to a fixed plane and intersecting a curve in the plane is called a **cylinder.** Any one of the lines is called an **element** of the cylinder.

80. Equations of cylinders

Consider the equation

$$x + y = 8. \tag{1}$$

Observe that no matter what number k represents, all the points $(2,6,k)$, $(4,4,k)$, and $(m,8-m,k)$ satisfy (1). But $(m,8-m,k)$ represents every point on a line through $(m,8-m,0)$ and perpendicular to the xy-plane. Hence, the locus of (1) is the plane

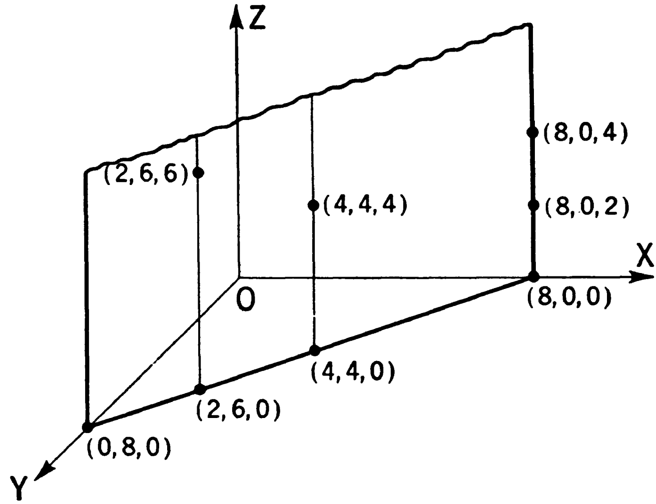


FIG. 3.

(see Figure 3) perpendicular to the xy-plane and containing the line in the xy-plane whose plane equation is $x + y = 8$.

Again, consider the equation

$$x^2 + y^2 = 25. \qquad (2)$$

No matter what k is, the point $(a,\pm\sqrt{25-a^2},k)$ satisfies (2). Hence, the locus of (2) is the cylinder consisting of all lines perpendicular to the xy-plane and intersecting the circle in the xy-plane represented by the plane equation $x^2 + y^2 = 25$. Figure 4 represents the locus.

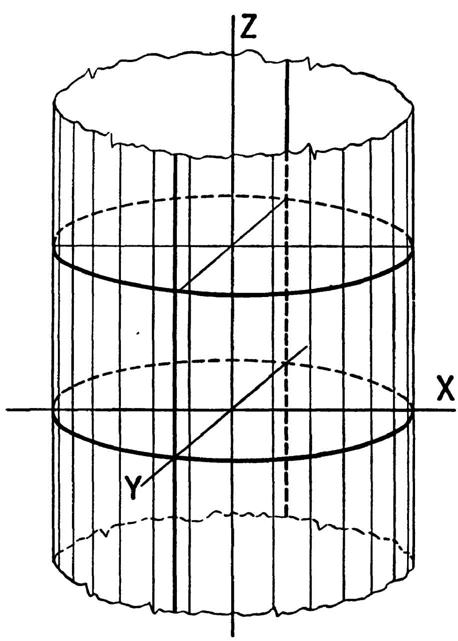


FIG. 4.

It extends upward and downward without bound.*

Figure 5 is represented by the equation

$$y^2 = 4z. \tag{3}$$

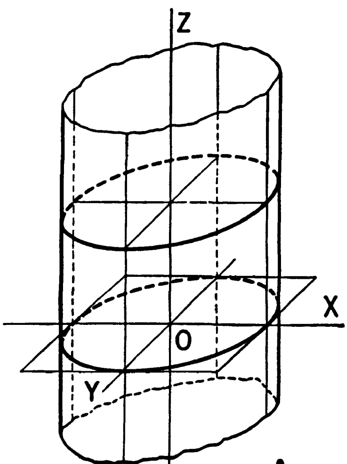
To sketch (3), we first sketch two parabolas with plane equations $y^2 = 4z$, one in the yz-plane and the other in a parallel plane, and then add lines to suggest the surface generated by moving the curve rightward. The actual surface extends without bound rightward, leftward, and upward.

The type of argument employed above may be used to show that *any equation*

$$f(x,y) = 0 \tag{4}$$

represents a cylinder with elements perpendicular to the xy-*plane and intersecting the curve in the* xy-*plane represented by the plane equation* $f(x,y) = 0$. Like statements apply to surfaces represented by $f(x,z) = 0$ and $f(y,z) = 0$. To plot a surface such as (4), we first *draw a curve, as in plane analytic geometry, in the plane of the variables, then a congruent parallel curve in a plane*

* The sketches in this book representing surfaces are designed to emphasize essential features and give clear ideas of general appearance without bringing in confusing details. The beginner easily learns to draw such figures, and the emphasis is on essentials important in analytic geometry. Figure A shows the draftsman's more exact *oblique drawing* of Figure 4, and Figure B is an *oblique drawing* of the paraboloid of Figure 8. Observe the elliptic and circular sections appearing as ellipses inscribed in parallelograms with sides parallel to the coordinate axes. The general plan of oblique drawing is often effective in sketching the part of a surface in the first octant.



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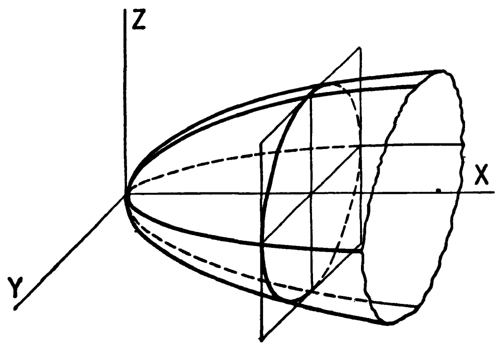


B

parallel to the first, and finally draw some lines parallel to the axis of the missing variable and extending from one curve to the other.

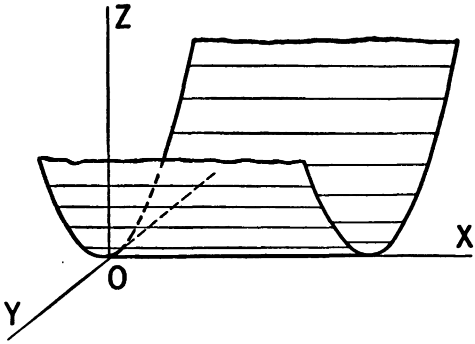


FIG. 5.

Exercises

1. Sketch the surfaces: (a) $x = 5$. (b) $y = 7$. (c) $z = -6$.

In Exercises 2 to 5, an equation and some points are given. State in each case which points of the group lie on the surface:

2. $x + y + z = 10$; A (1,2,8); B (3,−4,9); C (−5,−6,21).

3. $z^2 + x^2 + y^2 = 49$; A (3,2,6); B (3,−2,−6); C (7,1,−1); D (5,4,3).

4. $x^2 - y^2 = 12$; A $(4,2,k)$; B $(\sqrt{12},0,k)$; C $(0,-\sqrt{12},k)$; D $(-4,-2,-10^{10})$.

5. $4x^2 + y^2 = 8$; A $(-1,2,k)$; B $(2,k,0)$; C $(0, \sqrt{8},k)$; D $(k,2\sqrt{2-k^2},m)$.

Each of the surfaces numbered 6 to 9 contains the indicated point. Find the values of k:

6. $x + 2y + kz = 10$, (1,2,5).
7. $z + 2y - 2x = k$, (5,0,4).
8. $x^2 - ky^2 = 48$, $(4,-4,m)$.
9. $x^2 - 4y^2 - kz^2 = 10$, (1,2,3).

Sketch the surfaces represented by the equations numbered 10 to 15:

10. $x + y = 4$.
11. $2x + y = 5$.
12. $y + z = 4$.
13. $x + z = 6$.
14. $2x - y = 4$.
15. $y - 2z = 4$.

Sketch the surfaces represented by the equations numbered 16 to 23:

16. $x^2 + y^2 = 16$.
17. $x^2 + 4y^2 = 4$.
18. $z^2 = 2x$.
19. $y^2 = 4x$.
20. $x^2 - y^2 = 4$.
21. $y^2 = 4 - z$.
22. $x^2 + y^2 + 2x = 0$.
23. $y^2 + 4z^2 + 2y = 0$.

To describe a line, give a point on it and a line to which it is parallel, thus: the line through (2,3,0) parallel to the z-axis. Describe the lines

of intersection of the following pairs of surfaces numbered 24 to 32.

24. $x^2 + y^2 = 8, x = 2.$

25. $x + y = 5, y = 3.$

26. $y^2 + z^2 = 25, z = 3.$

27. $y^2 = 2z, z = 8.$

28. $x^2 - z^2 = 13, z = 6.$

29. $x^2 + z^2 - 4z = 0, z = 3.$

30. $x + y = 7, x - y = 11.$

31. $x^2 + z^2 = 5, x + z = 3.$

32. $x^2 - z^2 = 3, 2x - z = 3.$

81. Intercepts. Traces. Symmetry

To find the intercepts of a surface on the x-axis, set y and z equal to zero in its equation, and solve the result for x. The intercepts on the y-axis and the z-axis are found by a like method. For example, to find the intercepts of

$$x^2 + 2x + y^2 + 2z^2 = 8 \tag{5}$$

on the x-axis, set y and z equal to zero in (5) and solve the result for x to get $x = 2$ and $x = -4$. The intercepts on the y-axis are $\pm\sqrt{8}$, and on the z-axis ± 2.

The **trace** *of a surface on a plane is the intersection of the surface and the plane.* In the process of finding the shape of a surface from its equation, traces on planes parallel to the coordinate planes are basic. The trace of a surface on a plane parallel to a coordinate plane is specified by two equations, one the equation of the plane, the other the equation obtained by replacing one variable of the given equation by its value from the equation of the plane. Thus, the equations of the trace

of $x + y + z = 2$ on the xy-plane are:

$$z = 0, \; x + y + 0 = 2,$$

of $x - 2y + 4z = 6$ on the yz-plane are:

$$x = 0, \; 0 - 2y + 4z = 6,$$

of $x^2 + y^2 + z^2 = 12$ on $z = 2$ are:

$$z = 2, \; x^2 + y^2 + 4 = 12,$$

of $x^2 + 2x + y^2 - z^2 = 0$ on $z = 5$ are:

$$z = 5, \; x^2 + 2x + y^2 - 25 = 0.$$

The same general principles of determining the symmetry of a curve in plane analytic geometry apply to a surface in space. The most useful rule for elementary cases follows. *A surface is symmetric with respect to: the* xy*-plane if* z *occurs to even powers only*

in its equation, to the xz-*plane if* y *occurs to even powers only, and to the* yz-*plane if* x *occurs to even powers only.* For example, the surface $x^2 + 2x + 3y^2 + 2z^2 = 8$ is symmetric to the xy-plane and to the xz-plane.

Example. Discuss the surface $x^2 + y^2 = 2z + 4$ as regards symmetry, intercepts, and traces on the coordinate planes and on planes parallel to the xy-plane.

Solution. The surface is symmetric with respect to the xz-plane and the yz-plane.

From the given equation, we obtain $x = \pm 2$ by letting y and $z = 0$, $y = \pm 2$ by letting x and $z = 0$, and $z = -2$ by letting x and $y = 0$.

The equations of the trace on the xz-plane, obtained by letting $y = 0$ in the given equation, are

$$y = 0, \; x^2 = 2z + 4. \tag{a}$$

The equations of the trace on the yz-plane are

$$x = 0, \; y^2 = 2z + 4. \tag{b}$$

The equations of the trace on the plane $z = k$ are

$$z = k, \; x^2 + y^2 = 2k + 4. \tag{c}$$

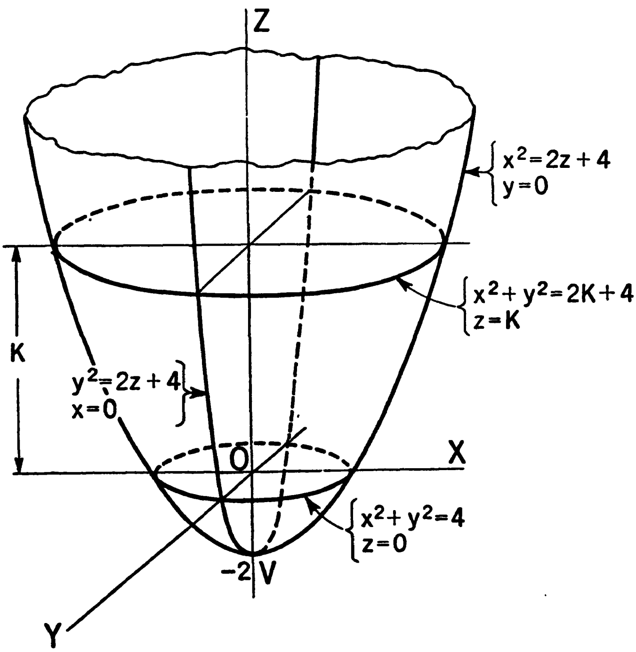


FIG. 6.

Observe that these traces defined by (c) are circles of radius $\sqrt{2k+4}$. If $k = 0$, we get from (c) a circle of radius 2 on the xy-plane; if $k = -2$, we get the point circle $x^2 + y^2 = 0$; if $k < -2$, there is no trace. Figure 6 represents the surface.

Exercises

For the equations numbered 1 to 6, find the intercepts on the axes, the equations of the traces on the coordinate planes, and the planes of symmetry.

1. $2x + y - 3z = 6$.

2. $x^2 + y^2 + z^2 = 25$.

3. $x^2 + 4y^2 - z^2 = 64$.

4. $x^2 - y^2 = 2z - 4$.

5. $y^2 = 2z - 5$.

6. $x^2 - 4y^2 - 4z^2 = 25$.

For each of the equations numbered 7 to 14, sketch the traces on the coordinate planes and on the indicated planes:

7. $x + 2y + 3z = 6$.
Traces on $z = 2$, $x = 6$, and $y = 6$.

8. $x^2 + y^2 = 2z$.
Traces on $z = 2$ and $z = \frac{9}{2}$.

9. $x^2 + y^2 + 4z^2 = 4$.
Traces on $z = \frac{1}{2}$ and $z = 1$.

10. $x^2 + y^2 = 4 - 2z$.
Traces on $z = 2$ and $z = -\frac{5}{2}$.

11. $x^2 + y^2 - z^2 = 4$.
Traces on $z = -4$ and $z = 4$.

12. $x^2 - y^2 - z^2 = 9$.
Traces on $x = -5$, $x = -3$, $x = 3$, and $x = 5$.

13. $x^2 - y^2 = 4$.
Traces on $z = -2$, $z = 4$, and $z = 6$.

14. $y^2 + z^2 = 16$.
Traces on $x = 5$ and $x = 10$.

82. Quadric surfaces

A surface is generally sketched from an equation by drawing its traces on one or more coordinate planes and then fitting into

the picture curves representing cross sections by a set of planes parallel to a coordinate plane (see Figure 6). The main types of surfaces represented by equations of the second degree in x, y, and z will now be considered.

The **ellipsoid** is represented by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{6}$$

From the equation, we see that the intercepts are

$$x = \pm a,\ y = \pm b,\ z = \pm c.$$

The surface is symmetric to all three coordinate planes. Also note that

$$|x| \leqq a,\ |y| \leqq b,\ |z| \leqq c.$$

Replacing z by k in (6), and writing it in the form

$$\frac{x^2}{a^2(1 - k^2/c^2)} + \frac{y^2}{b^2(1 - k^2/c^2)} = 1, \tag{7}$$

we see that the cross section cut out by $z = k$ is an ellipse. Figure 7 is a sketch of the surface.

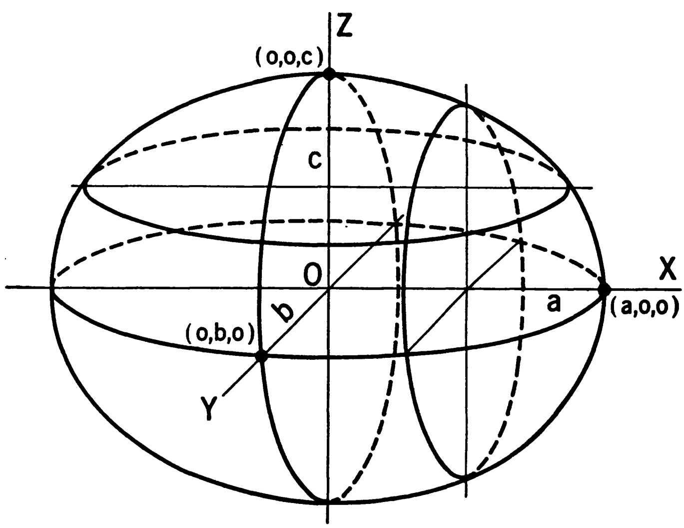
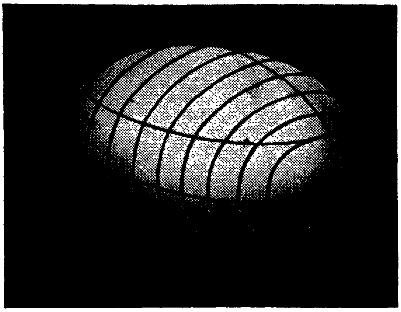


FIG. 7. Ellipsoid.

It is interesting to observe that a sphere may be regarded as a special case of an ellipsoid for which $a^2 = b^2 = c^2$. Also, an ellipsoid having two of the quantities a^2, b^2, and c^2 equal is called an *ellipsoid of revolution*, for it may be obtained by revolving an ellipse about one of its axes.



ELLIPSOID

The **elliptic paraboloid** is represented by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz. \tag{8}$$

The discussion is left to the student. Observe that the cross section cut out by $z = k$ is an ellipse

$$\frac{x^2}{a^2ck} + \frac{y^2}{b^2ck} = 1. \tag{9}$$

Figure 8 represents the surface.

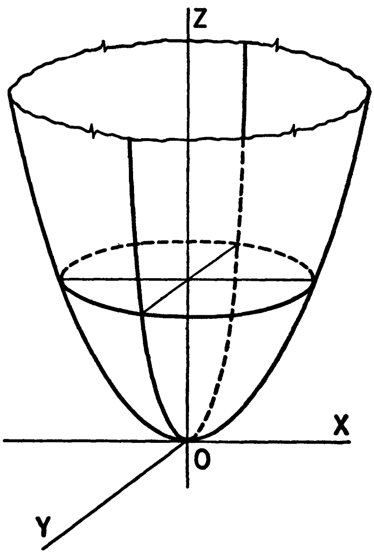
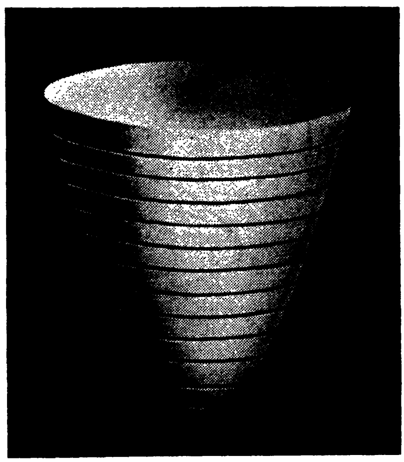


FIG. 8. Elliptic Paraboloid.



ELLIPTIC PARABOLOID

The **hyperboloid of one sheet** is represented by (see Figure 9)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \tag{10}$$

The discussion is left to the student. The cross section cut out by $z = k$ is an ellipse represented by

$$\frac{x^2}{a^2(1 + k^2/c^2)} + \frac{y^2}{b^2(1 + k^2/c^2)} = 1. \tag{11}$$

This ellipse exists for any value of k and is smallest when $k=0$.

The **hyperboloid of two sheets** is represented by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \tag{12}$$

The discussion is left to the student. The cross section cut out by $x = k$ is represented by

$$\frac{y^2}{b^2[(k^2/a^2) - 1]} + \frac{z^2}{c^2[(k^2/a^2) - 1]} = 1. \tag{13}$$

If $|k| > a$, Equation (13) represents a real ellipse; if $|k| < a$, (13) represents no real curve; and if $|k| = a$, (13) represents the point $(a,0,0)$. Figure 10 on page 218 represents the surface.

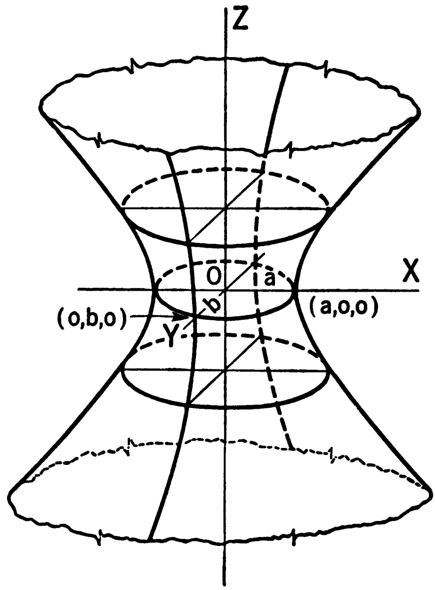
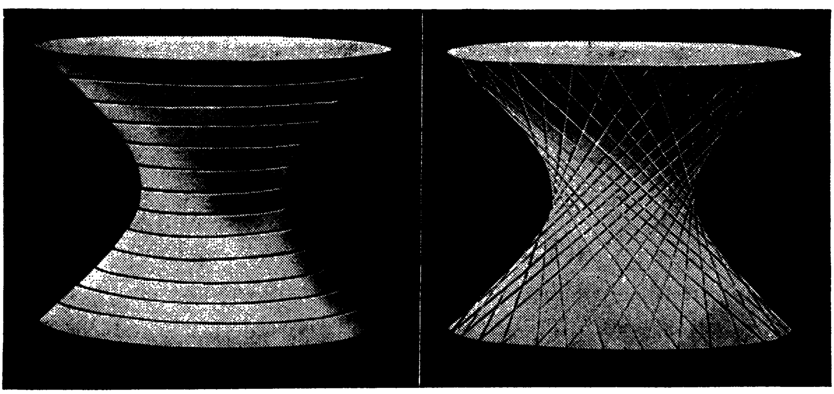


FIG. 9. Hyperboloid of One Sheet.



HYPERBOLOIDS OF ONE SHEET

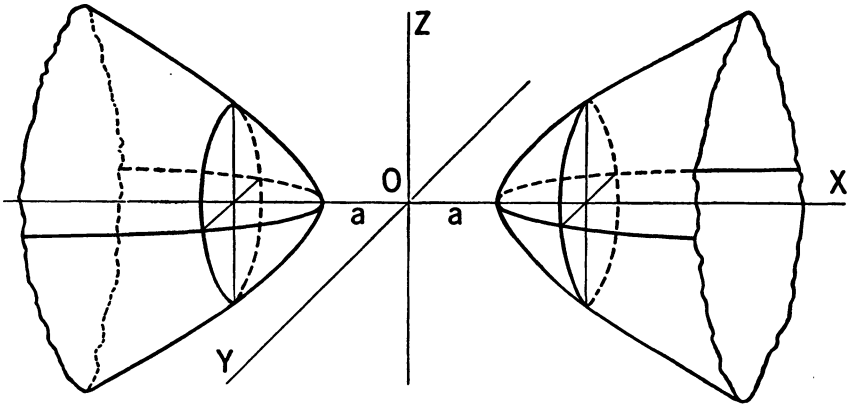


FIG. 10. Hyperboloid of Two Sheets.

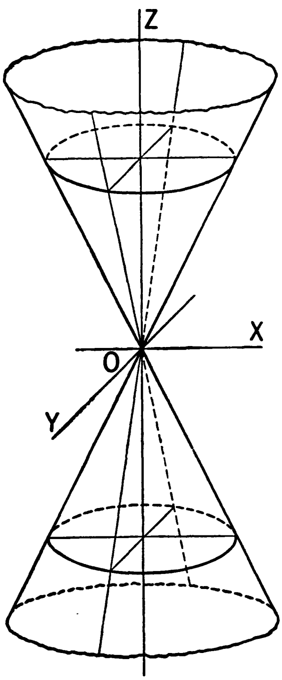


FIG. 11.

The **cone** represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \qquad (14)$$

may be considered a degenerate hyperboloid. The traces on the xz-plane are the lines

$$y=0, \frac{x^2}{a^2}-\frac{z^2}{c^2}=0; \text{ or } y=0, \frac{x}{a}=\pm\frac{z}{c}. \qquad (15)$$

Figure 11 represents the surface.

The **hyperbolic paraboloid** is represented by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = cz. \qquad (16)$$

It has no intercepts except zero, and it is symmetric with respect to the xz-plane and the yz-plane. The trace on the xz-plane is the parabola $x^2/a^2 = cz$, represented in Figure 12 by the curve AOB. The trace on the yz-plane is the parabola $-y^2/b^2 = cz$, represented in Figure 12 by curve EOF; and the traces on the xy-plane are the lines $(x^2/a^2) - (y^2/b^2) = 0$, represented by OC and OD in

Figure 12. The cross sections perpendicular to the z-axis are hyperbolas. Figure 12 represents the surface.

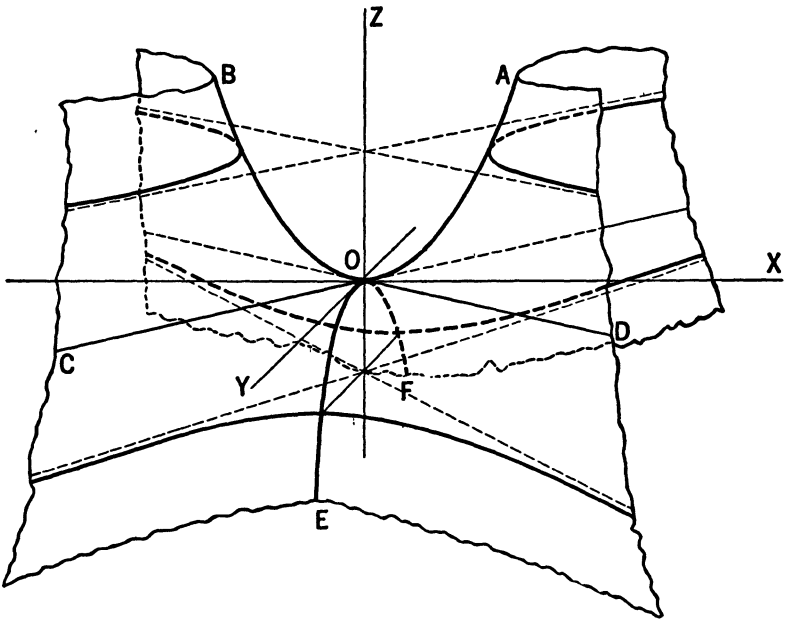
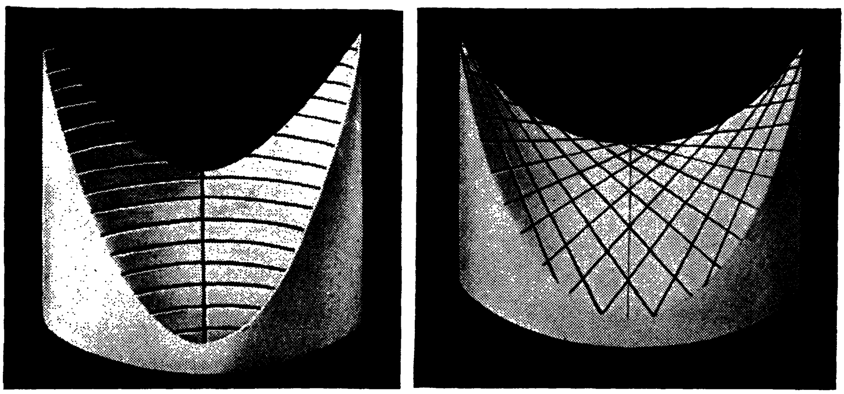


FIG. 12. Hyperbolic Paraboloid.



HYPERBOLIC PARABOLOIDS

Besides the surfaces already considered in this article, a second-degree equation may represent two planes, a single plane, a line, a point, an elliptic cylinder, a hyperbolic cylinder, or a parabolic cylinder, or no locus.

83. Other surfaces

A linear equation always represents a plane, as we shall show in the next chapter. It can be sketched by employing the plan used in §81, and the same statement applies to many other surfaces. Consider, for example, the part of

$$z = x\sqrt{25 - x^2 - y^2} \tag{17}$$

above the xy-plane. If $x = 0$, then $z = 0$ for all values of y. Hence, the complete y-axis lies on the locus. If $z = 0$, then $x^2 + y^2 = 25$, and the corresponding circle, besides the y-axis, is the trace in the xy-plane. The trace on the xz-plane is $z = x\sqrt{25 - x^2}$. In addition, the traces on planes $x = 1, 2, 3$, and 4 should be considered. Also note that $x^2 + y^2$ cannot be greater than 25, and that the surface is symmetric with respect to the xz-plane.

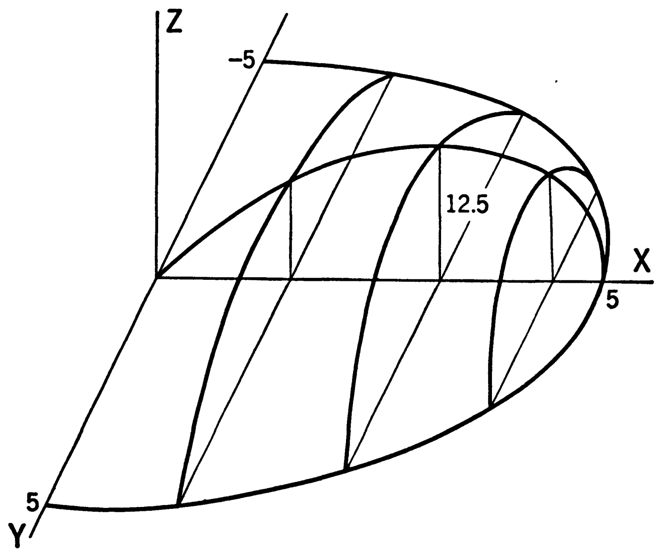


FIG. 13.

Figure 13 represents half of the locus of (17); there is a similar piece symmetric to it with respect to the y-axis.

Exercises

Name and sketch the surfaces represented by the equations numbered 1 to 9. Also name and give the dimensions of the traces on the indicated planes in each case.

1. $x^2 + y^2 + z^2 = 25.$ Traces on $x = 3$ and $x = -4$.

2. $x^2 + 4y^2 + 9z^2 = 36.$ Traces on $z = 0$, $y = 0$, and $x = 3$.

3. $x^2 + 4y^2 - z^2 = 4.$ Traces on $z = 0$ and $z = 2$.

4. $x^2 - 4y^2 - z^2 = 4.$ Traces on $x = 2$ and $x = -4$.

5. $-x^2 + 4y^2 + z^2 = 0.$ Traces on $x = -2$ and $x = 4$.

6. $x^2 + y^2 - 4z = 0.$ Traces on $z = 2$ and $z = -2$.

7. $y^2 + 4z^2 = -4x.$ Traces on $x = -4$ and $x = 4$.

8. $4x^2 - y^2 + 4z^2 = 16.$ Traces on $y = 2$ and $y = -4$.

9. $y^2 + z^2 - x^2 + 16 = 0.$ Traces on $x = 1$ and $x = 5$.

Sketch the surface represented by each equation:

10. $x^2 - y^2 = 2z.$

11. $4x^2 - y^2 = 2z.$

12. $x + y + z = 4.$

13. $x + 2y + 3z = 6.$

14. $x - 2y + 3z = 6.$

★15. $4x^2 + 3y^2 = 12yz.$

★16. $x^2 = 2yz.$

★17. $z^2 = 2xy.$

★18. $xy = z.$

★19. $x^2 = z + y.$

Sketch and name the following surfaces:

20. $x^2 + y^2 = 1.$

21. $x^2 + y^2 = 0.$

22. $x^2 + y^2 = x.$

23. $x^2 + y^2 = z.$

24. $x^2 + y^2 = z^2.$

25. $x^2 + y^2 = z^2 + 1.$

26. $x^2 + y^2 = z^2 - 1.$

27. $x^2 + y^2 = 1 - z^2.$

28. $x^2 + y^2 = z^2 + 2z.$

29. $x^2 + y^2 = z^2 + 2z + 1.$

84. Translation of axes

Figure 14 represents an original set of axes OX, OY, OZ, and a parallel set $O'X'$, $O'Y'$, $O'Z'$, where point O' is the point (a,b,c) referred to the original axes. P is a point (x,y,z) referred to the original axes and a point (x',y',z') referred to the new set. From the figure, we see that

$$\begin{aligned} x &= OC = AB = a + x' \\ y &= CF = DE = b + y' \\ z &= FP \qquad\quad\;\; = c + z' \end{aligned}$$

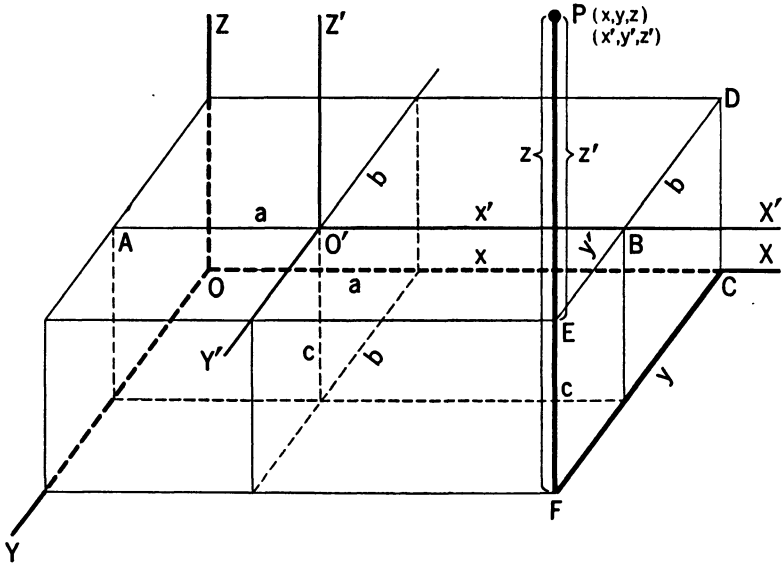


FIG. 14.

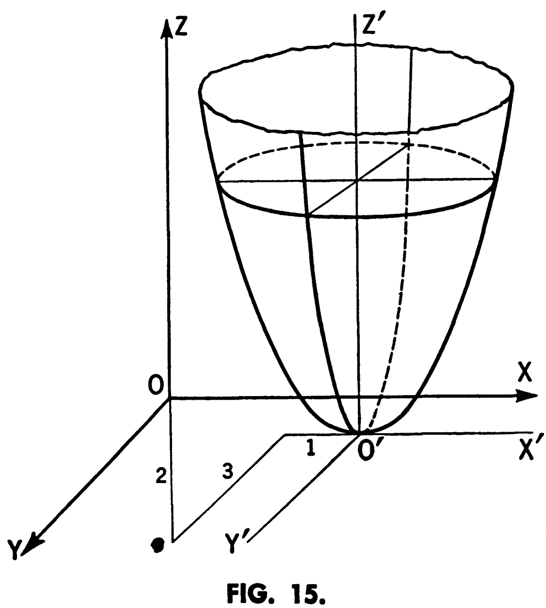


FIG. 15.

or, collecting the formulas for convenient reference:

$$x = x' + a,\ y = y' + b,\ z = z' + c. \tag{18}$$

A simple equation of a surface can often be obtained by using (18) to refer it to a set of axes parallel to the original set. For example, the surface

$$(x-1)^2 + (y+3)^2 = z+2$$

takes the form

$$x'^2 + y'^2 = z' \tag{19}$$

under the translation

$$x = x' + 1,\ y = y' - 3,\ z = z' - 2.$$

To obtain Figure 15, the surface (19) was sketched relative to the $x'y'z'$-axes. Both sets of axes are shown.

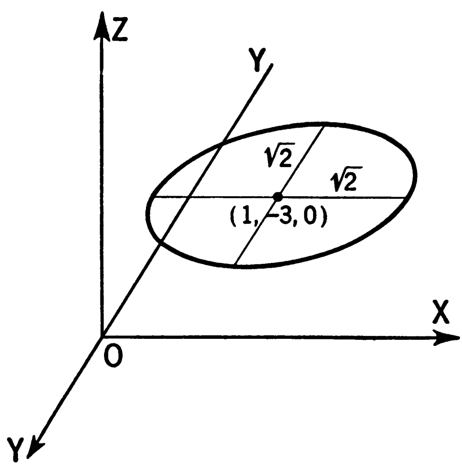


FIG. 16.

Figure 16 shows the trace of the surface on the xy-plane.

Exercises

For each of the surfaces represented by the following equations, obtain by translation of axes a simple equation, sketch the surface relative to the new axes, and show the original set of axes:

1. $(x-3)^2 + (y-4)^2 + (z-5)^2 = 4.$
2. $2(x-2)^2 + 2(y-3)^2 = z - 2.$
3. $4(x+2)^2 + 4y^2 = z^2.$
4. $x^2 + y^2 - (z+2)^2 = 4.$
5. $x^2 - (y-2)^2 - (z+2)^2 = 4.$
6. $x^2 + 2x + y^2 - z^2 - 2z = 4.$
7. $x^2 + y^2 - 4y - z^2 + 4 = 0.$

85. Equations of curves

A curve in space may be thought of as the intersection of two surfaces. It may be defined by any two equations both of which are satisfied by the points on the curve and both by no others. This idea has already been used in connection with traces. Thus, the trace of $x^2 + y^2 + z^2 = 25$ on the xy-plane is defined by

$$x^2 + y^2 = 25,\ z = 0, \tag{20}$$

and the circle is thought of as the intersection of a cylindrical surface and a plane.

Evidently, a curve may be defined by a great variety of pairs of equations, since there are infinitely many surfaces through it. For example, if a curve is represented by

$$f_1(x,y,z) = 0,\ f_2(x,y,z) = 0, \tag{21}$$

then the surface

$$f_1(x,y,z) + kf_2(x,y,z) = 0, \tag{22}$$

where k may be any constant, contains the curve (21). For, if any point (a,b,c) satisfies (21), the result of substituting it in (22) will be $0 + k0 = 0$, and this is true for all values of k. Any two equations obtained from (22) by assigning two distinct values to k may be used to represent the curve (21).* This idea is often employed to obtain equations of a curve by finding the equations of two cylinders each having elements perpendicular to a coordinate plane. For example, the curve defined by

$$x^2 + 2y^2 - z^2 = 4,\ x^2 + y^2 + z^2 = 25 \tag{23}$$

is contained in the surface

$$x^2 + 2y^2 - z^2 - 4 + k(x^2 + y^2 + z^2 - 25) = 0 \tag{24}$$

and may be represented by the two cylinders got by taking $k = 1$ and $k = -2$ in (24), namely,

$$2x^2 + 3y^2 - 29 = 0,\ -x^2 - 3z^2 + 46 = 0.$$

To obtain the equation of the surface projecting a curve onto the xy, xz, *or* yz*-plane, eliminate* z, y, *or* x, *respectively, from the equations of the curve.*

Example. Sketch the part in the first octant of the curve

$$x^2 + y^2 = z,\ 2x^2 + 2y^2 - 2x = z. \tag{a}$$

Solution. The curve may be represented by the first equation of (a) and the result of subtracting the first of (a) from the second, namely,

$$x^2 + y^2 = z,\ x^2 + y^2 - 2x = 0. \tag{b}$$

* It is assumed that $f_1(x,y,z)$ and $f_2(x,y,z)$ are single-valued functions; that is, to each point (x,y,z) there is associated at most one value of f_1 and one value of f_2.

Figure 17 shows the desired curve OB as the intersection of a paraboloid and a cylinder. The other part of the curve is a similar piece symmetric to curve OB with respect to the xz-plane.

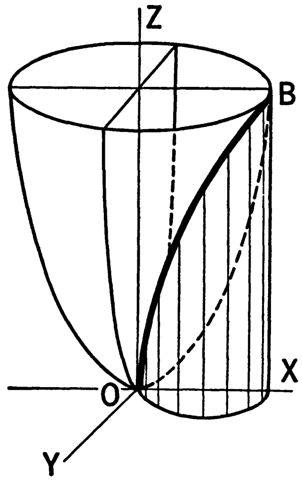


FIG. 17.

86. Intersections of three surfaces

The points common to three surfaces **may be found by solving their equations simultaneously. This idea will be used** in the solution of the following example.

Example. Show that the curves represented by $x + y + z + 2 = 0, x + y + 1 = 0$ and $2x^2 = yz, y^2 + 4xz = 0$ meet in a point.

Solution. We shall find the solutions of the first three given equations and test the results in the fourth one. From the first two given equations, we get $z = -1$. Substitute $z = -1$ and $y = -1 - x$ in the third equation to get

$$2x^2 = x + 1, \text{ or } x = 1,\ x = -\tfrac{1}{2}.$$

From $y = -1 - x$, $y = -2$ and $-\frac{1}{2}$ and the solutions are the coordinates of points $(1,-2,-1)$ and $(-\frac{1}{2},-\frac{1}{2},-1)$. By trial, $(1,-2,-1)$ is found to satisfy the fourth given equation, and the curves meet in $(1,-2,-1)$.

Exercises

1. Which of the points $(1,-1,-4)$, $(3,3,12)$, $(-3,5,28)$, $(2,0,-2)$, $(4,-2,-14)$, and $(-4,4,26)$ lie on the curve $x^2 + y^2 - z = 6$, $2x - 4y + z = 2$?

2. Find a point on the line $5x + y + z = 10$, $x + 2y - z = 2$ having as x-coordinate: (a) $x = 0$; (b) $x = 1$; (c) $x = 2$; (d) $x = 3$; (e) $x = m$.

Sketch the part in the first octant of each curve defined by each of the pairs of equations numbered 3 to 12:

3. $x + y = 4, y + z = 4.$

4. $x^2 + y^2 = 25, z = 3.$

5. $x^2 + y^2 = 25, x = 3.$

6. $x^2 + y^2 = 25, y^2 + z^2 = 25.$

7. $x^2 + y^2 = z$, $x = y$.

8. $xy + xz + yz = 1$, $z = 0$.

9. $(x - 1)^2 + y^2 = 1$, $x^2 + y^2 = z$.

10. $x^2 + y^2 + z^2 = 25$, $16x^2 + 16y^2 = 9z^2$.

11. $x^2 + y^2 = z$, $x + y = 2$.

12. $x^2 + z^2 = 1$, $y = 3x^2 + 4z^2$.

13. Find the point where the line $x + y + z = 10$, $2x - y + 2z + 6 = 0$ meets: (a) the xy-plane; (b) the plane $z = 1$.

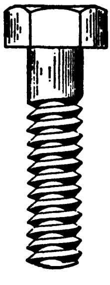


FIG. 18.

14. Find the points where the line $x + 2y + z = 2$, $x + z - 1 = 0$ meets the paraboloid $2x^2 + z^2 = 2y$.

15. Show that the lines $z = -2$, $x + y + z + 3 = 0$ and $2x + y = 1$, $x - y + z = 3$ meet in a point.

16. Using an equation having form (22) for the curve $x^2 + y^2 = 5$, $z = x$, find the equation of a surface through the curve and having: (a) 5 as x-intercept; (b) passing through (1,2,1); (c) having as trace on the yz-plane $y^2 + 5z - 5 = 0$.

17. A helix is a curve like that defined by threads of a bolt (see Figure 18), or like the cut edges of the drill shown in Figure 19.

(a) If its equations are

$$x = a \cos kz, \quad y = a \sin kz, \qquad \text{(A)}$$

show that the helix lies on the cylinder

$$x^2 + y^2 = a^2. \qquad \text{(B)}$$

FIG. 19.

(b) If P is any point on the helix (A) (see Figure 20), Q the projection of P on the xy-plane, and M the point where curve (A) crosses the x-axis, show that

(circular arc MQ)/QP= ak.

(c) If the cylinder (B) is cut along the element MN (see Figure 20) and spread out flat in a plane, show that the curve (A) will appear as a series of parallel line segments, and find the tangent of the angle made by each of the lines with MN.

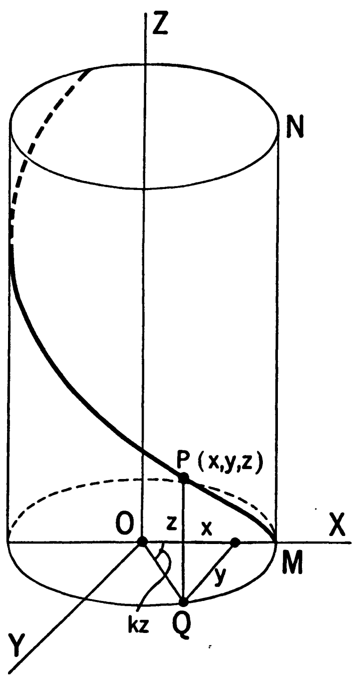


FIG. 20.

CHAPTER XIII

Lines and Planes in Space

87. Distance between two points. Direction cosines

Figure 1(a) shows two points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ with plane m_1 through P_1 and plane m_2 through P_2, both perpen-

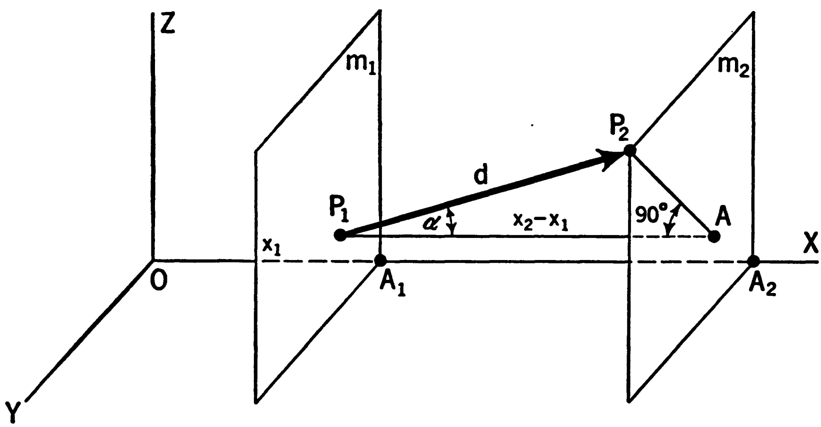


FIG. 1. (a)

dicular to the x-axis, m_1 meeting it in A_1 and m_2 meeting it in A_2. Then,

$$\overrightarrow{OA_2} = \overrightarrow{OA_1} + \overrightarrow{A_1A_2}, \text{ or } \overrightarrow{A_1A_2} = \overrightarrow{OA_2} - \overrightarrow{OA_1}, \tag{1}$$

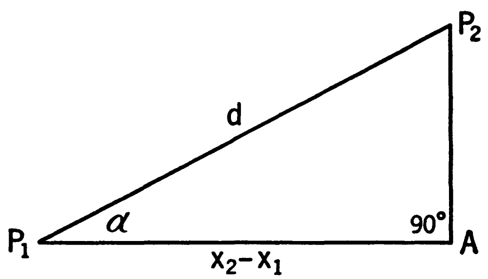


FIG. 1. (b)

and, since the x-coordinate of a point (x,y,z) is the vector from O to the point $(x,0,0)$,

$$\overrightarrow{OA}_1 = x_1, \ OA_2 = x_2. \tag{2}$$

From (2) and (1),

$$\overrightarrow{A_1A_2} = x_2 - x_1. \tag{3}$$

From P_1 draw a line perpendicular to m_2 meeting it in A. Then,

$$\overrightarrow{P_1A} = \overrightarrow{A_1A_2} = x_2 - x_1. \tag{4}$$

Denoting by α the angle AP_1P_2 and by d the length $|\overrightarrow{P_1P_2}|$ from P_1 to P_2, we obtain from the right triangle P_1AP_2 with right angle at A (see Figures 1(a) and 1(b)),

$$\cos \alpha = (x_2 - x_1)/d. \tag{5}$$

Figure 2 represents the same points P_1 and P_2 with planes drawn through them perpendicular to the coordinate axes to

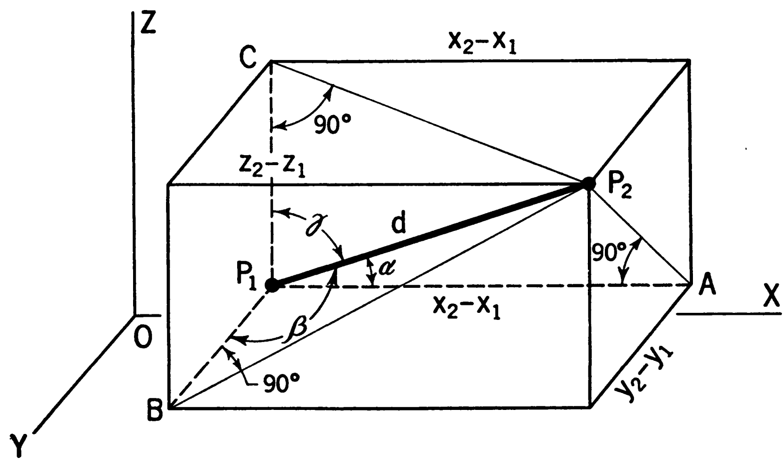


FIG. 2.

form the rectangular parallelepiped shown. Since the dimensions of the parallelepiped are $|x_2 - x_1|$, $|y_2 - y_1|$, and $|z_2 - z_1|$, we have

$$d = |P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}, \tag{6}$$

since the diagonal of a rectangular parallelepiped is equal to the square root of the sum of the squares of its edges.

The angles which two directed lines in space make with each other are defined to be the plane angles which intersecting lines having the same directions, respectively, make with each other. The angles α, β, and γ, which a line P_1P_2 directed from P_1 to P_2, or vector $\overrightarrow{P_1P_2}$, makes with the x-axis, the y-axis, and the z-axis, respectively, are called the **direction angles** of $\overrightarrow{P_1P_2}$, and $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the **direction cosines** of $\overrightarrow{P_1P_2}$. If two vectors have the same direction, evidently they have the same direction cosines; if they have opposite directions, their direction angles differ by 180° and their respective direction cosines are numerically equal but opposite in sign.

Denoting by α, β, and γ the direction angles of the vector $\overrightarrow{P_1P_2}$ in Figure 2, we read from the right triangles P_1AP_2, P_1BP_2, and P_1CP_2,

$$\begin{aligned} \cos\alpha &= (x_2 - x_1)/d, \\ \cos\beta &= (y_2 - y_1)/d, \\ \cos\gamma &= (z_2 - z_1)/d.^* \end{aligned} \tag{7}$$

The numbers (7), $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the **direction cosines** of vector $\overrightarrow{P_1P_2}$. The numbers (7), or the results of multiplying the right members of (7) by -1, are also called direction cosines of line P_1P_2.

Replacing in (7) x_1, y_1, and z_1 each by zero, and x_2, y_2, and z_2 by x, y, and z, respectively, and changing slightly, we have *for the vector* $\overrightarrow{\mathrm{OP}}$ *from the origin to any point in space,*

$$x = d\cos\alpha,\ y = d\cos\beta,\ z = d\cos\gamma. \tag{8}$$

From (7) we obtain

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}{d^2},$$

or, replacing d^2 by its value from (6),

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{9}$$

---

* These formulas hold for any angles made by $\overrightarrow{P_1P_2}$ with the positive directions of the axes. However, we generally think of the angles no greater numerically than 180°.

In words, *the sum of the squares of the direction cosines of any line in space is* 1.

If all three direction cosines of a line are multiplied by a number k different from zero, the results are called **direction numbers** of the line. If l, m, and n are direction numbers of a line, then there is a number k such that

$$l = k \cos \alpha, \; m = k \cos \beta, \; n = k \cos \gamma. \tag{10}$$

Then

$$l^2 + m^2 + n^2 = k^2(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) = k^2 \tag{11}$$

because of (9). Since, from (10), the direction cosines are l/k, m/k, and n/k, we see from (11) that *direction cosines of a line are obtained from direction numbers by dividing the direction numbers by the square root of the sum of their squares.* Thus, if 2, 3, and -6 are direction numbers of a line, its direction cosines are $2/\sqrt{4 + 9 + 36} = \frac{2}{7}, \frac{3}{7}, -\frac{6}{7}$ or $-\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$.

If two lines are parallel, their direction cosines are equal and their direction numbers, being multiples of direction cosines, must be proportional.

Example. Find the distance from $P_1(1,3,7)$ to $P_2(0,5,5)$ and the direction cosines of P_1P_2.

Solution. Using (6) and (7), we obtain

$$d = \sqrt{(0 - 1)^2 + (5 - 3)^2 + (5 - 7)^2} = 3.$$

$$\cos \alpha = \frac{0 - 1}{3} = -\frac{1}{3}, \quad \cos \beta = \frac{5 - 3}{3} = \frac{2}{3},$$

$$\cos \gamma = \frac{5 - 7}{3} = -\frac{2}{3}.$$

Exercises

1. Find the distances P_1P_2, given:

(a) $P_1(3,-4,6)$, $P_2(5,-2,8)$.

(b) $P_1(-4,2,-7)$, $P_2(-6,-1,-8)$.

(c) $P_1(1,-2,0)$, $P_2(0,2,-8)$.

(d) $P_1(3,1,-3)$, $P_2(1,-1,-5)$.

2. Find the direction cosines of the vectors $\overrightarrow{P_1P_2}$ of Exercise 1.

3. Find the perimeter of the triangle ABC with:

(a) $A(-1,2,3)$, $B(0,-2,11)$, $C(2,1,6)$; (b) $A(3,-1,0)$, $B(2,-5,8)$, $C(0,-1,4)$.

4. Show that the triangle ABC is a right triangle if:
(a) $A(1,0,-3)$, $B(3,4,7)$, $C(5,8,-7)$; (b) $A(0,0,3)$, $B(2,-3,9)$, $C(12,-8,-5)$.

5. Show by distances that points $(1,-2,5)$, $(-1,-5,-1)$, and $(3,1,11)$ lie in a straight line.

6. What is the distance between two planes, one through $(1,-2,-6)$ and the other through $(3,-4,7)$, if both are perpendicular to: (a) the x-axis? (b) the y-axis? (c) the z-axis?

7. Find the direction cosines of the line AB if:
(a) $A(2,3,-7)$, $B(3,-1,1)$; (b) $A(2,0,7)$, $B(-5,6,13)$.

8. Find direction cosines of a line having direction numbers:
(a) 2, 4, -4; (b) 1, -4, -8; (c) 1, 2, 3; (d) 6, 6, 7.

9. Give the direction cosines of: (a) the x-axis; (b) the y-axis; (c) the z-axis.

Hint. The x-axis passes through (0,0,0) and (1,0,0), or, for the x-axis, $\alpha = 0°$, $\beta = 90°$, $\gamma = 90°$.

10. There are two lines each consisting of points in the xy-plane equidistant from the x-axis and the y-axis. Find direction cosines of each line.

11. Find the direction cosines of a line from the origin to (x,y,z).

12. Find the direction cosines of the line having x,y, and z equal to each other for every point on it. What is the inclination of this line to each axis?

13. Find the missing direction cosine in each case: (a) $\cos\alpha = \frac{1}{2}$, $\cos\beta = \frac{1}{2}$; (b) $\cos\alpha = 0$, $\cos\beta = 0$.

14. Is there a line for which $\cos\alpha = 0$, $\cos\beta = 0$, and $\cos\gamma = 0$?

15. How is a line situated relative to an axis of coordinates if:
(a) $\cos\alpha = 0$? (b) $\cos\beta = 0$? (c) $\cos\gamma = 0$?

16. A line segment 6 units long has direction cosines $\frac{1}{9}$, $-\frac{4}{9}$, and $-\frac{8}{9}$. Find the distance between two parallel planes through the ends of this segment if they are perpendicular to: (a) the x-axis; (b) the y-axis; (c) the z-axis.

17. Find one positive direction cosine of a line perpendicular to: (a) the xy-plane; (b) the xz-plane; (c) the yz-plane.

Express each condition numbered 18 to 21 by means of an equation:

18. Point (x,y,z) is distant 1 from the origin.

19. Point (x,y,z) is on a sphere of center (1,2,0) and radius 3.

20. Point (x,y,z) is equidistant from (0,0,0) and $(3,-2,1)$.

21. Point (x,y,z) is a vertex of a right triangle having the ends of its hypotenuse at $(0,0,-2)$ and $(3,0,0)$.

22. Write two equations which must hold if (x,y,z), $(0,0,2)$, and $(3,1,0)$ lie in a straight line.

88. Angle which one directed line makes with another

If α, β, γ and α', β', γ' are the direction angles of two lines and θ an angle which one of the lines makes with the other, then

$$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'. \qquad (12)$$

The derivation of this equation follows. Let OP (see Figure 3) be a unit vector through the origin O with direction angles α, β,

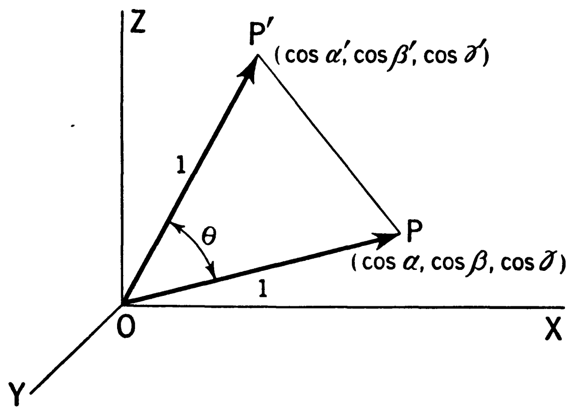


FIG. 3.

and γ and OP' a unit vector through O with direction angles α', β', and γ'. Then, in accordance with the definition of §**87**, angle POP' is an angle made by one of the given lines with the other. Applying Formulas (8), §**87**, with d equal to 1, we find that P is the point $(\cos\alpha, \cos\beta, \cos\gamma)$ and that P' is the point $(\cos\alpha', \cos\beta', \cos\gamma')$. Hence, applying the law of cosines to triangle POP' in Figure 3, we get

$$\overline{PP'}^2 = 1^2 + 1^2 - 2\cdot 1\cdot 1\cos\theta = 2 - 2\cos\theta, \qquad (13)$$

or, using the distance formula (6) for PP' in (13),

$$(\cos\alpha - \cos\alpha')^2 + (\cos\beta - \cos\beta')^2 + (\cos\gamma - \cos\gamma')^2 = 2 - 2\cos\theta.$$

Expanding the parentheses, using (9), §**87**, to simplify the result, and then solving for $\cos\theta$, obtain (12).

If l,m,n and l',m',n' are *direction numbers* of the given lines, (12) may be written

$$\cos\theta = \frac{ll' + mm' + nn'}{\pm\sqrt{l^2 + m^2 + n^2}\,\sqrt{l'^2 + m'^2 + n'^2}}. \tag{14}$$

In §**87** we saw that parallel lines have proportional direction numbers. Conversely, *if two lines have proportional direction numbers, they are parallel.* For, in this case, their respective direction cosines are equal or equal numerically and opposite in sign; in either case, the right member of (12) for them is, by (9), either $+1$ or -1 and θ is 0° or 180°; in other words, the lines are parallel.

If two lines are perpendicular, then θ in (12) is 90°, $\cos\theta = 0$, and

$$\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' = 0; \tag{15}$$

and, conversely, if (15) holds, the lines with direction angles α, β, γ and α', β', γ' are perpendicular. Also, if $\cos\theta = 0$ in (14), we may multiply through by the denominator of the right member to obtain

$$ll' + mm' + nn' = 0.$$

Hence, *if* l,m,n *are direction numbers of a line and* l',m',n' *are direction numbers of a perpendicular line, then*

$$\mathbf{ll' + mm' + nn' = 0}, \tag{16}$$

and conversely.

Example. Find the angle ABC of the triangle having its vertices at $A(1,-2,3)$, $B(2,4,-6)$, and $C(-3,1,-1)$.

Solution. The direction cosines of $\overrightarrow{BA}$ and $\overrightarrow{BC}$, by (7), are $-1/\sqrt{118}$, $-6/\sqrt{118}$, $9/\sqrt{118}$ and $-5/\sqrt{59}$, $-3/\sqrt{59}$, $5/\sqrt{59}$, respectively. Hence, by (12),

$$\cos\theta = \frac{(-1)(-5) + (-6)(-3) + (9)(5)}{\sqrt{118}\,\sqrt{59}} = \frac{68}{59\sqrt{2}},$$

and from this, angle θ is found to be **35.4°**.

Exercises

Each of the exercises numbered 1 to 5 gives two sets of direction numbers. Find the smallest positive angle between the corresponding lines:

1. $3,-4,2$ and $4,-3,0$.

2. $1,-2,2$ and $2,-1,2$.

3. $2,-3,6$ and $1,4,8$.

4. $3,4,2$ and $2,2,-3$.

5. $1,1,1$ and $-1,-1,-1$.

6. Are lines with direction numbers $2,4,-3$ and $-2,-4,3$ parallel?

Show that lines having direction numbers specified in Exercises 7 and 8 are mutually perpendicular:

7. $1,2,-2$; $0,1,1$; $4,-1,1$.

8. $-2,2,1$; $2,1,2$; $-1,-2,2$.

9. Show that three of the lines determined by pairs of the following points are mutually perpendicular: $(1,2,-3)$, $(2,6,5)$, $(9,-2,-2)$, $(-3,-5,1)$.

10. Solve the problem resulting from replacing the points in Exercise 9 by $(5,-2,0)$, $(12,4,6)$, $(5,4,-6)$, and $(-7,5,7)$.

11. Using (16), show that $(2,0,-3)$, $(5,6,-9)$, and $(8,3,3)$ are the vertices of a right triangle.

12. Find l and m if a line with direction numbers $l,m,3$ is perpendicular to two lines having direction numbers $1,2,2$ and $13,8,2$, respectively.

13. Find the direction angles of a line equally inclined to the three coordinate axes.

14. Find the equation of a plane parallel to the x-axis and passing through $(1,-2,4)$ and $(6,3,-6)$. Note that its equation does not contain x.

15. Three vertices of a parallelogram are $A(4,3,5)$, $B(0,6,0)$, and $C(-8,1,4)$. Find the fourth vertex: (a) if AC is a diagonal; (b) if AB is a diagonal; (c) if BC is a diagonal.

16. Find direction cosines of a line perpendicular to each of two lines having respective direction numbers: (a) $2,3,6$ and $3,4,12$; (b) $1,-3,2$ and $2,-1,-2$.

89. Equation of a plane

Let OE, directed from O to E, be a line through the origin O and having direction angles α, β, and γ. Let $P_1(x_1,y_1,z_1)$ be any point on line OE (see Figure 4). We seek the equation of the plane through P_1 and perpendicular to OE.

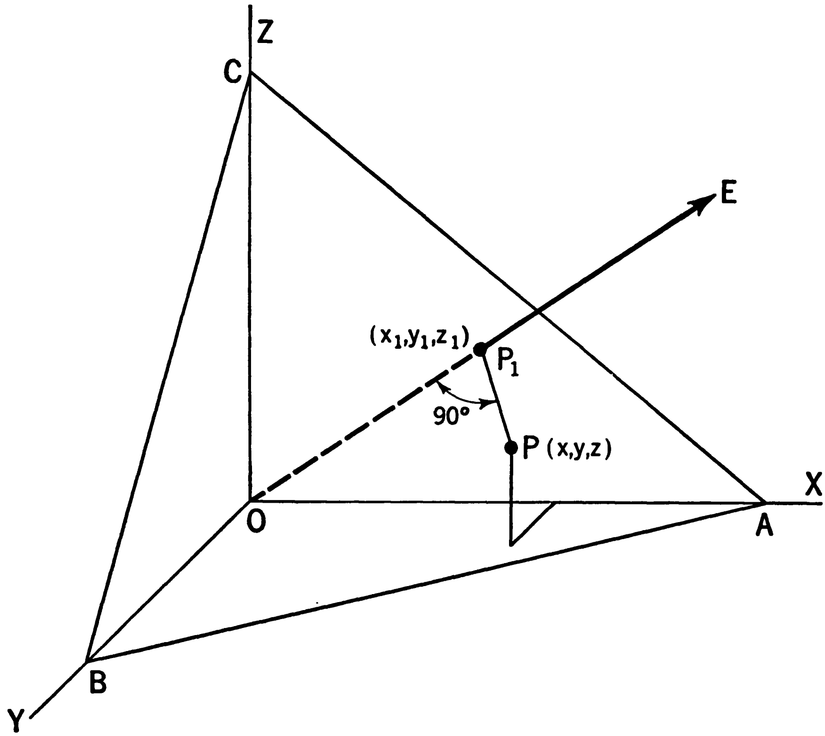


FIG. 4.

Let p be the directed distance from O to P_1. Then, from (8), §87, we have

$$x_1 = p \cos \alpha,\ y_1 = p \cos \beta,\ z_1 = p \cos \gamma, \tag{17}$$

where p is positive or negative according as $\overrightarrow{OP_1}$ and $\overrightarrow{OE}$ have the same or opposite directions. Let $P(x,y,z)$ be any point in the required plane. Then PP_1 is perpendicular to OE. Direction numbers of PP_1 are $x - x_1, y - y_1, z - z_1$, and the direction cosines of OE are $\cos \alpha, \cos \beta, \cos \gamma$. Hence, by (16), §88,

$$(x - x_1) \cos \alpha + (y - y_1) \cos \beta + (z - z_1) \cos \gamma = 0, \tag{18}$$

or

$$x \cos \alpha + y \cos \beta + z \cos \gamma = x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma. \quad (19)$$

Replacing x_1, y_1, and z_1 in (19) by their values from (17), and replacing $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$ in the result by 1, we get

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p. \quad (20)$$

Any point P_2 not in the required plane could not satisfy (18), for OP_1P_2 would not be a right angle. Hence, (18), and therefore (20), is the required equation.

If O *is the origin and* OE *is a directed line having direction angles* α, β, *and* γ *and meeting a plane at right angles in* P, *the equation of the plane is*

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p, \quad (21)$$

where $|\overrightarrow{OP}| = |p|$ *and* p *is positive, negative, or zero according as* OP *has the same direction as* OE, *the direction opposite to* OE, *or is zero.*

A line perpendicular to a plane is often called a **normal** to the plane. When p is positive, (21) is called the **normal form** of the equation of a plane.*

If the numbers A, B, and C, not all zero, are thought of as direction numbers of a line, then the linear equation

$$\frac{Ax + By + Cz}{\sqrt{A^2 + B^2 + C^2}} = \frac{D}{\sqrt{A^2 + B^2 + C^2}} \quad (22)$$

has the form (21) and, therefore, is the equation of a plane. The result of clearing (22) of fractions represents the same plane. Hence, comparing (22) with (21), we deduce that *the general linear equation*

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C}z = \mathrm{D} \quad (23)$$

represents a plane distant $|\mathrm{D}/\sqrt{\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2}|$ *from the origin and perpendicular to a line having direction numbers* A, B, *and* C.

The normal form of Equation (23) is obtained by dividing it through by $\pm\sqrt{A^2 + B^2 + C^2}$, where the sign of the radical is

* If $p = 0$, the normal form requires that the rightmost non-zero coefficient in (21) be positive.

the same as the sign of D if $D \neq 0$, the same as that of C if $D = 0$ and $C \neq 0$, and the same as that of B if $D = 0$ and $C = 0$ and $B \neq 0$.

Example 1. Write in normal form the equation of the plane $6x + 6y - 7z = 33$. Find the distance from the origin to the plane and the direction cosines of a normal to it.

Solution. Dividing the given equation by $\sqrt{6^2 + 6^2 + 7^2} = 11$, we obtain

$$\tfrac{6}{11}x + \tfrac{6}{11}y - \tfrac{7}{11}z = 3.$$

The distance from the origin to the plane is **3** and the direction cosines of a normal to it are **6/11, 6/11,** and **−7/11.**

Example 2. Find the equation of a plane through the points (0,1,0), (2,−2,1), and (1,0,1).

Solution. Let the equation of the plane be

$$ax + by + cz = d. \tag{a}$$

Since each point lies on the plane, its coordinates must satisfy (a). Hence,

$$\begin{aligned} 0 + b + 0 &= d, \\ 2a - 2b + c &= d, \\ a + 0 + c &= d. \end{aligned}$$

The solution of these equations is $a = 2d$, $b = d$, $c = -d$. Replacing a, b, and c in (a) by these values, and dividing through by d, obtain

$$\mathbf{2x + y - z = 1.}$$

Exercises

Write each of the equations numbered 1 to 6 in normal form, and find direction cosines of a normal to the plane represented and the distance of this plane from the origin:

1. $2x - 2y - z = 12$.

2. $6x - 6y - 7z + 55 = 0$.

3. $3x + 4y + 25 = 0$.

4. $7x + 6 = 0$.

5. $3x - 4y = 0$.

6. $x - 6y - 2z = 0$.

7. Give direction numbers of a line to which both of the planes $x + y + z = 10$ and $x - y - 2z = 20$ are parallel.

Hint. The required line is perpendicular to a normal to each plane.

8. Show that the normals to the planes $x + y - 3z = 12$ and $3x + 3y + 2z = 0$ are perpendicular.

9. Find an acute angle between a normal to $x + 2y - 2z = 10$ and a normal to $x + 4y - 8z = 36$.

10. Find the equation of a plane through $(1,2,-3)$ and perpendicular to a line with direction numbers 6,6,7.

11. Find the equation of a plane if a perpendicular to it from the origin meets it in $(2,3,-6)$.

12. Find the intercepts of the plane $2x - 2y + z = 10$ on the axes.

13. Describe the planes represented by:

(a) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

(b) $(x - x_1) \cos \alpha + (y - y_1) \cos \beta + (z - z_1) \cos \gamma = 0.$

(c) $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0.$

For each set of three points numbered (14) to (17), find the equation of a plane containing the three points:

14. $(0,0,1)$, $(0,2,0)$, $(1,-1,1)$.

15. $(1,1,1)$, $(2,0,-1)$, $(0,1,2)$.

16. $(1,-3,3)$, $(0,-2,2)$, $(-4,5,-4)$.

17. $(0,0,0)$, $(0,1,2)$, $(-3,4,1)$.

18. Prove that the following equation represents a plane through points (x_1,y_1,z_1), (x_2,y_2,z_2), and (x_3,y_3,z_3):

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

19. Find the equation of the plane bisecting at right angles the line segment connecting $(1,-2,4)$ to $(3,2,-6)$.

20. Find the coordinates of the point where a perpendicular from the origin to plane $6x + 6y + 7z = 10$ meets the plane.

90. Angle between two planes. Parallel and perpendicular planes

The angle between two planes is the angle between their normals. Direction numbers of the normals to the planes

$$\begin{aligned} Ax + By + Cz &= D \\ A'x + B'y + C'z &= D' \end{aligned} \tag{24}$$

are A,B,C and A',B',C', respectively. Obtaining the direction

cosines from these by dividing by the respective square roots of the sum of their squares, and using (12), §88, we obtain

$$\cos\theta = \frac{\pm(AA' + BB' + CC')}{\sqrt{A^2 + B^2 + C^2}\sqrt{A'^2 + B'^2 + C'^2}}. \tag{25}$$

If the planes are perpendicular, $\cos\theta$ in (25) is zero. Hence, *the condition that planes* (24) *be perpendicular is*

$$\mathbf{AA' + BB' + CC' = 0.} \tag{26}$$

Two planes are parallel if their normals are parallel. Since the direction numbers of the normals to planes (24) are A,B,C and A',B',C', and lines are parallel if their direction numbers are proportional, planes (24) are parallel if

$$A' = kA,\ B' = kB,\ C' = kC,\ k \neq 0. \tag{27}$$

Conversely, if the planes (24) are parallel, their normals are parallel and (27) holds. Hence, *planes* (24) *are parallel if, and only if, equations of the form* (27) *hold.*

Example 1. Find the equation of a plane parallel to $x - 2y - 2z = 12$ and passing through $(2,-1,3)$.

Solution. An equation of the form

$$x - 2y - 2z = D \tag{a}$$

is parallel to the given plane. Since $(2,-1,3)$ is to lie on plane (a), we have

$$2 - 2(-1) - 2(3) = D, \text{ or } D = -2. \tag{b}$$

Hence, the required equation is $\boldsymbol{x - 2y - 2z = -2}$.

91. Distance from a plane to a point

Let any plane be represented in form (21), §89, by

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p. \tag{28}$$

Transforming (28) by the translation to parallel axes (see Figure 5) with origin (x_1,y_1,z_1),

$$x = x' + x_1,\ y = y' + y_1,\ z = z' + z_1, \tag{29}$$

we get

$$(x' + x_1)\cos\alpha + (y' + y_1)\cos\beta + (z' + z_1)\cos\gamma = p,$$

or

$$x' \cos \alpha + y' \cos \beta + z' \cos \gamma = p - x_1 \cos \alpha - y_1 \cos \beta - z_1 \cos \gamma. \quad (30)$$

Now (28) and (30) represent the same plane, and the distance from the new origin (x_1,y_1,z_1) to plane (30), and therefore to (28),

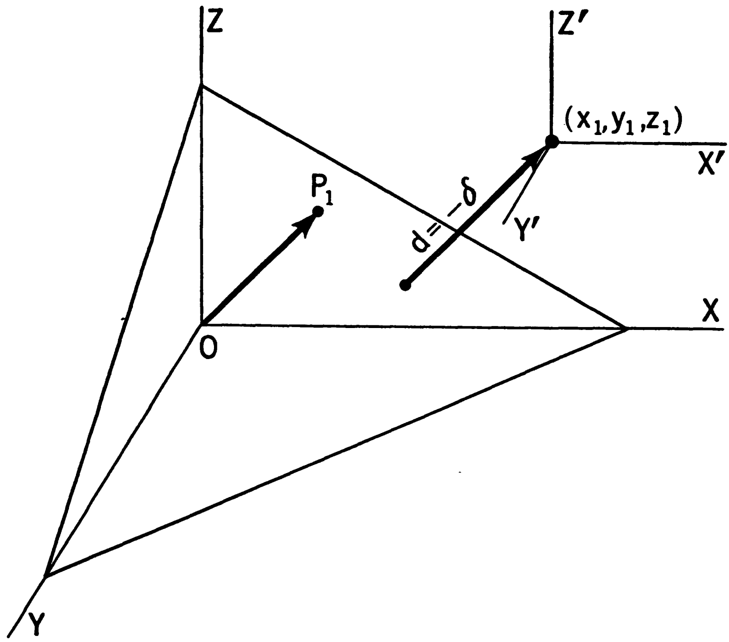


FIG. 5.

is the right member of (30). Therefore, *the distance* δ *from* (x_1,y_1,z_1) *to plane* (28) is given by

$$\delta = p - (x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma). \quad (31)$$

The distance d *from plane* (28) *to point* (x_1,y_1,z_1), being the negative of δ in (31), *is given by*

$$\mathbf{d = x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p}. \quad \mathbf{(32)}$$

This formula shows that *the directed distance from a plane to a point* (x_1,y_1,z_1) *may be found by writing the equation of the plane in normal form, transferring all terms to the left member, and sub-*

stituting x_1 *for* x, y_1 *for* y, *and* z_1 *for* z *in the result.* Applying this procedure to the plane

$$Ax + By + Cz = D, \tag{33}$$

we obtain for *the distance* d *from plane* (33) *to* (x_1,y_1,z_1)

$$d = \frac{Ax_1 + By_1 + Cz_1 - D}{\pm\sqrt{A^2 + B^2 + C^2}},$$

where the sign before the radical is the same as that of D. For example, the distance from $2x + 2y - z = 24$ to point $(5,-6,3)$ is given by

$$d = \frac{2(5) + 2(-6) - (3) - 24}{\sqrt{2^2 + 2^2 + (-1)^2}} = -\frac{29}{3}.$$

The minus sign indicates that $(5,-6,3)$ is on the same side of the plane as the origin.

Exercises

1. Which pairs of the following equations represent parallel planes and which pairs represent perpendicular planes?

(a) $3x + 4y - z = 0.$ (d) $2z - 6x - 8y = 9.$

(b) $4x - 2y + 4z = 3.$ (e) $3x - 2y + 2z = 4.$

(c) $y - 2x - 2z = 5.$ (f) $2x + 2y - z = 10.$

Find the angles between the pairs of planes numbered 2 to 4:

2. $2x - y + 2z = 10,\ x - 2y - 2z = 5.$

3. $x - 4y + 8z = 12,\ 8x - 4y + z = 10.$

4. $x + 2y - 3z = 10,\ 2x - y + 6z = 8.$

5. Find k if planes $2x + y - z = 12$ and $6x + ky - 3z = 0$ are parallel.

6. Find k if planes $x + ky + 2z = 0$ and $3x + 3y - kz = 10$ are perpendicular.

Find the equation of each plane satisfying one of the sets of conditions numbered 7 to 11:

7. Parallel to $x + y + 3z = 8$ and passing through $(2,-1,0)$.

8. Perpendicular to a line with direction numbers $2,2,-1$ and passing through $(2,-3,4)$.

9. Passing through $(0,0,-2)$ and $(1,1,1)$ and perpendicular to plane $x - y + z = 10.$

10. Perpendicular to $2x + y - z = 10$ and to $x - 2y + z = 5$, and passing through (0,0,0).

11. Perpendicular to $x - y - 2z = 12$ and to $x + y = 7$ and passing through (1,0,2).

Describe the systems of planes represented by the equations numbered 12 to 15:

12. $x + y + z = k$.

13. $x \cos \alpha + y \cos \beta + z \cos \gamma = 5$.

14. $ax + by = c$.

15. $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$.

Find the distance from each of the planes numbered 16 to 19 to the associated point:

16. $2x + 2y - z = 12$; $(1,-3,2)$.

17. $x + 2y - 2z = 13$; $(2,4,1)$.

18. $x + 4y - 8z = 55$; $(7,8,-1)$.

19. $x + y = 3\sqrt{2}$; $(\sqrt{2}, 5\sqrt{2}, 0)$.

20. Find the equations of two planes bisecting the dihedral angles formed by planes $2x + 6y - 3z = 12$ and $6x - 3y + 2z = 15$.

21. The base of a pyramid lies in plane $4x - 7y + 4z = 18$ and its vertex is $(6,7,-3)$. Find the length of the altitude of the pyramid.

22. The coordinate planes and a plane having x-intercept 3, y-intercept 4, and z-intercept 6, bound a tetrahedron. (a) Find the area of the oblique face by equating two expressions for the volume. (b) Show that the square of the area of the oblique face equals the sum of the squares of the areas of the other three faces.

23. In Exercise 22, replace the intercepts 3,4, and 6 by a,b, and c, respectively, and solve the resulting problem.

24. Find the volume of a pyramid having its vertices at (1,0,0), (0,0,2), (0,3,0), and $(6,9,-8)$.

92. Equations of lines

Two non-parallel planes intersect in a line. Since the points on the line are common to the two planes, their coordinates must satisfy the equations of both planes. Hence, two linear equations represent a line. Consider, for example, the two equations

$$\begin{aligned} x + y - z &= 12, \\ x - y + 2z &= 6. \end{aligned} \tag{34}$$

Solving these for x and y in terms of z, we get

$$x = 9 - \tfrac{1}{2}z, \; y = 3 + \tfrac{3}{2}z. \tag{35}$$

For each value of z we get values for x and y, that is, the coordinates of a point on line (34). Thus, if $z = 0$, we obtain from (35) $x = 9$, $y = 3$, and (9,3,0) is on line (34); if $z = 2$ in (35), $x = 8$, $y = 6$, and (8,6,2) is on line (34); and it appears that for each value of z we get a point on line (34).

If (x_1,y_1,z_1) is a fixed point on a line having direction angles α, β, and γ, and if (x,y,z) is any point on the line, then from (7), §87,

$$\frac{x - x_1}{d} = \cos\alpha, \; \frac{y - y_1}{d} = \cos\beta, \; \frac{z - z_1}{d} = \cos\gamma,$$

or solving each equation for d and equating its three values,

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}. \tag{36}$$

These are called the **symmetric equations of a line.** The ratios obtained by multiplying each denominator by k will still be equal; that is, direction cosines of the line may be replaced by direction numbers. Hence, *a line through* (x_1,y_1,z_1) *and having direction numbers* l, m, *and* n *is represented by*

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.* \tag{37}$$

Equating the ratios of (37) to r and solving for x, y, and z, we obtain

$$x = x_1 + rl, \; y = y_1 + rm, \; z = z_1 + rn. \tag{38}$$

These equations are called **parametric equations of the line** having direction numbers l, m, and n and passing through (x_1,y_1,z_1).

The equations of a line through two points (x_1,y_1,z_1) and (x_2,y_2,z_2) are easily obtained by using (37) with the direction numbers l, m, and n replaced by $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$. The result is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \tag{39}$$

* If l, m, or n is zero, form (37) involves division by zero and cannot be used. Suppose $n = 0$; then $\cos\gamma = 0$ and $\gamma = 90°$. Hence, the line is perpendicular to the z-axis and z is constant. Therefore, when $n = 0$, (37) may be replaced by $z = z_1$, $(x - x_1)/l = (y - y_1)/m$. A similar statement applies if $l = 0$ or $m = 0$.

Example. Write in three forms the equations of a line through $(1,-2,0)$ and having direction numbers 1,4,8.

Solution. Using the given numbers in (37) and in (38), we get

$$\frac{x-1}{1}=\frac{y+2}{4}=\frac{z}{8}; \tag{a}$$

$$x=1+r,\ y=-2+4r,\ z=8r. \tag{b}$$

Replacing 1, 4, and 8 in (a) by $\frac{1}{9}$, $\frac{4}{9}$, and $\frac{8}{9}$, we get

$$\frac{x-1}{1/9}=\frac{y+2}{4/9}=\frac{z}{8/9}. \tag{c}$$

93. Line defined by two linear equations

A line may be represented by the equations of any two planes through it. An illustration will show a method of getting desired information concerning a line so represented. If a line is represented by

$$9x-6y-4z+12=0,\ 3x+3y-3z-11=0, \tag{40}$$

it may, in accordance with §**85**, be represented by any two equations of the system represented by

$$9x-6y-4z+12+k(3x+3y-3z-11)=0. \tag{41}$$

Taking k in (41) equal to -3 and to 2, we obtain

$$\begin{aligned} -15y+5z+45&=0,\\ 15x-10z-10&=0. \end{aligned} \tag{42}$$

Note that these equations represent the line (40) and that the first represents a plane perpendicular to the yz-plane and the second a plane perpendicular to the xz-plane. Solving the first equation of (42) for z in terms of y and the second for z in terms of x, and changing the results slightly, we get

$$\frac{x-\frac{2}{3}}{\frac{2}{3}}=\frac{y-3}{\frac{1}{3}}=\frac{z}{1}. \tag{43}$$

Comparing (43) with (37), we deduce that Equations (40) represent a line having direction numbers $\frac{2}{3},\frac{1}{3},1$ and passing through $(\frac{2}{3},3,0)$.

Another method of getting the equations of a line in one of the standard forms of §**92** is to find two points on the line and then

use (39). Also, the direction numbers of a line defined by two linear equations may be found by using the fact that the line is perpendicular to the normals to the planes of the two given equations.

Exercises

Each set of equations numbered 1 to 5 defines a line; find a point on each line, and a set of direction numbers for it:

1. $x/2 = (y - 3)/4 = (-2z + 6)/3$.

2. $x = 3 + 2r,\ y = 5 - 3r,\ z = -2 + 6r$.

3. $x - 2 = 2y - 6 = 3z - 5$.

4. $x + y - 7z + 1 = 0,\ 5x - y - 5z = 13$.

5. $2x - 3y + 2z = -2,\ 2x + y - 6z = 14$.

6. Find the point where the line of Exercise 4 pierces: (a) the xy-plane; (b) the yz-plane. Write the equations of the line in the form (39).

7. Locate two points on the line given by $x + y + z - 6 = 0$, $2x - y - 2z + 5 = 0$, and then write the equations of the line in the form (39).

8. Find direction numbers of the line of Exercise 5. Let $l,m,1$ be the required direction numbers. Then $2l - 3m + 2 = 0$ and $2l + m - 6 = 0$. Why?

Find equations of each line satisfying each of the sets of conditions numbered 9 to 14:

9. Perpendicular to $x + y - z = 6$ and passing through (2,3,0).

10. Passing through $(1,2,-2)$ and $(6,4,-3)$.

11. Direction numbers 1,3,4 and passing through $(2,-4,0)$.

12. Passing through $(2,-1,3)$ and having $\cos\alpha = \cos\beta = \frac{1}{3}$.

13. Parallel to the line (43) and passing through the origin.

14. Perpendicular to two lines having respective direction numbers 4,1,2 and $3,-1,1$, and passing through $(1,0,-2)$.

Read the footnote relating to (37) and then write equations of the lines defined by the sets of conditions numbered 15 to 23:

15. Perpendicular to the lines $x = y = z$ and $x/2 = y/2 = -z$ and passing through their point of intersection.

16. Direction numbers 1,2,0, passing through $(1,-1,3)$.

17. Direction numbers 0,2,5, passing through (2,1,0).

18. Direction numbers 2,0,0, passing through (2,0,2).

19. Parallel to the x-axis and passing through $(1,-1,2)$.

20. Perpendicular to the z-axis, making an angle of 30° with the x-axis, and passing through $(1,0,-3)$.

21. Meeting the z-axis at right angles and passing through $(1,2,-3)$.

22. Parallel to the z-axis and passing through $(5,4,-2)$.

23. Perpendicular to the y-axis, parallel to plane $x + y - z = 0$, and passing through $(2,4,-3)$.

24. Write the conditions that a line with direction numbers l,m, and n be:

(a) Parallel to plane $ax + by + cz = d$.

(b) Perpendicular to plane $ax + by + cz = d$.

25. Find the angle made by the line $x/2 = y/3 = z/-6$ with the plane $x - 8y + 4z = 5$.

★26. Find the distance from the point $(7,2,3)$ to the line $(x - 1)/2 = (y - 2)/2 = z/(-1)$.

27. Find the equation of a plane determined by point $(-2,7,5)$ and line

$$2x - y + 3z = 10, \quad x + y + 2z = 12.$$

Hint. Any plane through the given line is given by

$$2x - y + 3z - 10 + k(x + y + 2z - 12) = 0.$$

Find the equation of each plane passing through the line of Exercise 27 and satisfying one of the conditions numbered 28 to 33:

28. Having x-intercept -2.

29. Parallel to the z-axis.

30. Perpendicular to the yz-plane.

31. Perpendicular to the plane $2x - y - z = 5$.

32. Parallel to the line $(x - 1)/5 = y/2 = (z + 2)/3$.

★33. Making an angle $\sin^{-1} (1/\sqrt{6})$ with the x-axis.

94. Miscellaneous problems

The following examples will illustrate some instructive methods.

Example 1. The curve $z = 2x + 3$, $y = 0$ is revolved about the x-axis. Find the equation of the surface generated.

Solution. Let $P(x,y,z)$ be any point on the surface, and pass a plane through P perpendicular to the x-axis. It must cut the surface under consideration in a circle (see Figure 6)

with center on the x-axis. If $(x_1,0,z_1)$ is the point where this circle cuts the xz-plane, z_1 is the radius of the circle. Since the radius of this circle is also $\sqrt{y^2 + z^2}$, and $x = x_1$ for all of its points,

$$x = x_1, \ z_1 = \pm\sqrt{y^2 + z^2}. \tag{44}$$

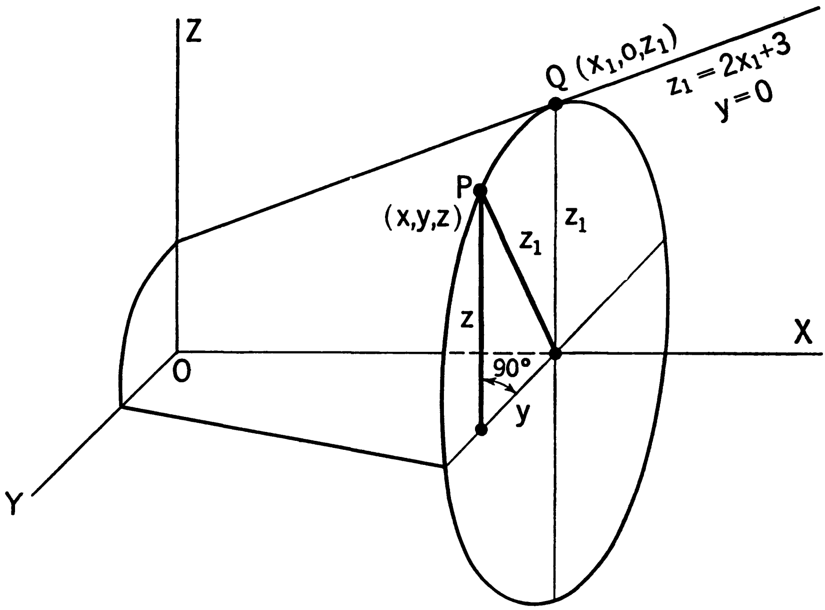


FIG. 6.

Since $Q(x_1,0,z_1)$ lies on line $z = 2x + 3$, $y = 0$, we have

$$z_1 = 2x_1 + 3. \tag{45}$$

Replacement of x_1 and z_1 in (45) by their values from (44) gives

$$\pm\sqrt{y^2 + z^2} = 2x + 3,$$

or

$$y^2 + z^2 = (2x + 3)^2.$$

A similar argument can be used to find the equations of surfaces of revolution about any line parallel to a coordinate axis.

Example 2. Prove that there is an infinite system of lines lying on the hyperboloid of one sheet; in other words, verify that it is a ruled surface.

Solution. The equation of the hyperboloid of one sheet has the form

$$l^2x^2 + m^2y^2 - n^2z^2 = a^2. \tag{a}$$

Transferring m^2y^2 to the right member, and factoring both members of the result, obtain

$$(lx - nz)(lx + nz) = (a - my)(a + my). \qquad \text{(b)}$$

If we let $lx - nz = k(a - my)$ in (b), we get

$$lx - nz = k(a - my), \; lx + nz = (1/k)(a + my). \qquad \text{(c)}$$

If a point satisfies both equations (c), it satisfies (a). Therefore, the system of lines (c) corresponding to various values of k lies on the surface (b) and therefore on (a).

Exercises

Find the equations of the surfaces generated by revolving the curves of Exercises 1 to 6 about the indicated lines:

1. $z = x^2$, $y = 0$; about the x-axis.

2. $z = x$, $y = 0$; about the z-axis.

3. $y^2 = 3x^2 + 4$, $z = 0$; about the x-axis.

4. $y^2 = 4x$, $z = 0$; about the y-axis.

5. $x = 2y + 3$, $z = 0$; about $x = 3$, $z = 0$.

6. $(z - 2)^2 = 4x$, $y = 0$; about $z = 2$, $y = 0$.

Find the equations of the loci of points satisfying conditions numbered 7 to 14:

7. Distance from point (0,0,2) equals distance from xy-plane.

8. Distance from x-axis equals distance from yz-plane.

9. Distance from x-axis equals distance from y-axis.

10. Distance from (0,0,3) is k times distance from (0,0,−3).

11. Lying on lines through the origin making a constant angle with the z-axis.

12. Twice as far from the z-axis as from the xy-plane.

13. Twice as far from plane $x + 2y = 2$ as from the origin.

14. Twice as far from plane $4x + 4y + 7z = 7$ as from plane $x + 4y + 8z = 9$.

15. Prove that the hyperbolic paraboloid

$$m^2x^2 - n^2y^2 = z$$

is a ruled surface.

16. If each of the letters, u, v, w, and s represents an expression of the form $ax + by + cz + d$, show that $uv = ws$ represents a ruled surface.

CHAPTER XIV

Cylindrical, Spherical, and Polar Coordinates

95. Cylindrical coordinates

Let $P(x,y,z)$ be any point in space and Q its projection on the xy-plane (see Figure 1). Then polar coordinates r,* θ of Q,

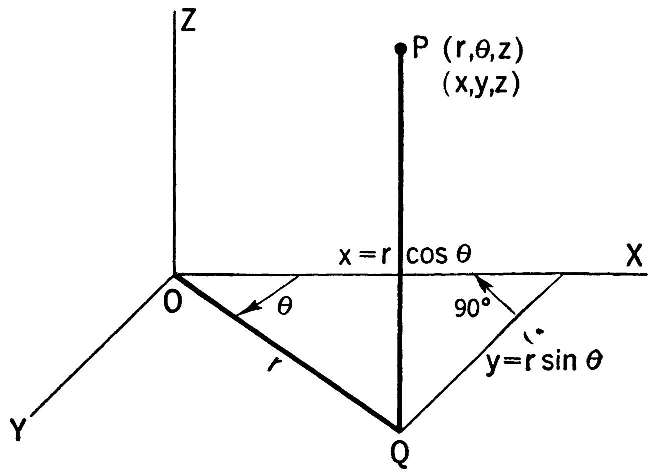


FIG. 1.

referred to OX as polar axis and O as pole, and z are the **cylindrical coordinates** of P, and P is designated by (r,θ,z). The relations

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z \tag{1}$$

connecting rectangular coordinates and cylindrical coordinates are easily derived.

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* r is used here instead of ρ because we wish to reserve ρ, in space, for the distance OP.

Example 1. Find cylindrical coordinates of (2,2,6) and rectangular coordinates of (4,60°,6).

Solution. The point (2,2) in plane rectangular coordinates is $(2\sqrt{2},45°)$ in plane polar coordinates. Therefore, (2,2,6) is the same point as $(\mathbf{2\sqrt{2},45°,6})$. For (4,60°,6) we get, from (1), $x = 4 \cos 60° = 2$, $y = 4 \sin 60° = 2\sqrt{3}$, $z = 6$. Therefore, (4,60°,6) is the same point as $(\mathbf{2,2\sqrt{3},6})$.

Example 2. Find the equation in cylindrical coordinates of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution. From (1), $x^2 + y^2 = r^2$. Using this in the given equation, we obtain

$$\mathbf{r^2 + z^2 = a^2.}$$

Exercises

1. Find cylindrical coordinates of the following points given in rectangular coordinates:

(a) (3,3,5). (c) $(2,-2\sqrt{3},6)$. (e) (1,0,0).

(b) (3,−3,2). (d) $(-2,2\sqrt{3},6)$. (f) (0,2,0).

2. Find rectangular coordinates of the points:

(a) (5,30°,3). (b) $(-7\sqrt{2},45°,10)$. (c) (5,90°,−5).

Describe and sketch the cylinders represented by the equations numbered 3 to 8:

3. $r = 1$. **5.** $r \sin \theta = 1$. **7.** $\theta = 30°$.

4. $r = a \cos \theta$. **6.** $r \cos (\theta + 60°) = 1$. **8.** $\theta = 120°$.

Transform the equations numbered 9 to 14 to equations in cylindrical coordinates:

9. $x^2 + y^2 - 4z^2 = 2$. **12.** $x^2 + y^2 = 2az$.

10. $z^2 - x^2 - y^2 = 2$. **13.** $z = xy$.

11. $x + y + z = 5$. **14.** $x^2 - y^2 = 3z$.

Sketch the surfaces represented by the equations numbered 15 to 20:

15. $r = z \cos \theta$. **17.** $r = 2z^2$. **19.** $r^2 \cos 2\theta = z$.

16. $r = 3z$. **18.** $r^2 + 4z^2 = 4$. **20.** $r^2 = z \sin 2\theta$.

96. Spherical coordinates

Let $P(x,y,z)$ be any point in space, Q its projection on the xy-plane (see Figure 2), and O the origin. Then ρ $(= OP)$, θ (the

angle OQ makes with the x-axis), and ϕ (the angle OP makes with the z-axis), are the **spherical coordinates** of P; and P is designated by (ρ,θ,ϕ).* The following relations between rectangular and spherical coordinates are easily derived from Figure 2:

$$x = \rho \sin \phi \cos \theta,$$
$$y = \rho \sin \phi \sin \theta, \qquad (2)$$
$$z = \rho \cos \phi.$$

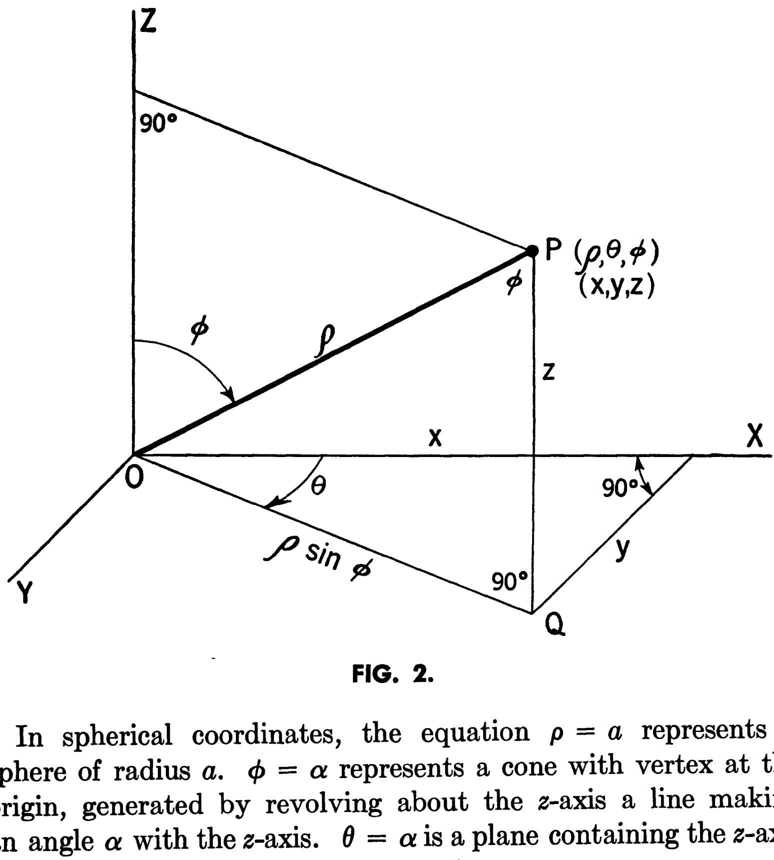

FIG. 2.

In spherical coordinates, the equation $\rho = a$ represents a sphere of radius a. $\phi = \alpha$ represents a cone with vertex at the origin, generated by revolving about the z-axis a line making an angle α with the z-axis. $\theta = \alpha$ is a plane containing the z-axis and making an angle α with the x-axis.

Example. Sketch $\rho = a \sin \phi \cos^2 \theta$.

Solution. The surface is symmetric to the yz-plane, and the discussion and figure refer only to the right half of it. The trace

* Because points on the earth's surface are located by means of two angles at the center of the earth, similar to the angles of spherical coordinates, θ is called the longitude and ϕ the co-latitude of P.

on the xy-plane, obtained by setting $\phi = 90°$ in the given equation, is shown as curve OAB in Figure 3. The section cut

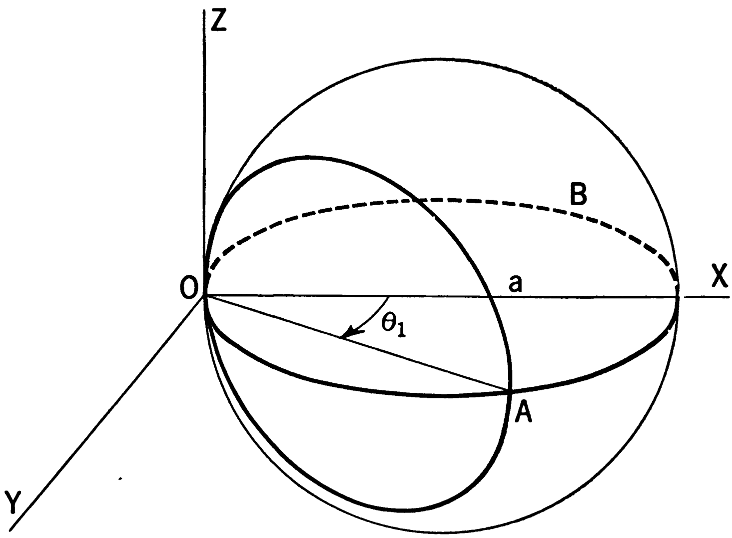

FIG. 3.

out by the plane $\theta = \theta_1$ is the circle $\rho = a\ (\cos^2 \theta_1) \sin \phi$. Hence, the sections by planes through the z-axis are circles, their diameters being chords, through the origin, of curve OAB.

97. Polar coordinates in space

If $P(x,y,z)$ is a point in space distant ρ from the origin, and if the direction angles of OP are α, β, and γ, then ρ, α, β, and γ are the space polar coordinates of P.

Evidently,

$$x = \rho \cos \alpha,$$
$$y = \rho \cos \beta, \tag{3}$$
$$z = \rho \cos \gamma.$$

These coordinates are subject to the condition (9), §87,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

Example. Describe the graphs of: (a) $\rho = 5$; (b) $\gamma = 30°$.

Solution. (a) The graph of $\rho = 5$ is the sphere with center at the origin and radius 5.

(b) Since γ is the angle that the line from the origin to a point P under consideration makes with the z-axis, we see that the locus is a cone with z-axis as axis, with vertex at the origin, and with angle between z-axis and element 30°.

Exercises

1. Find rectangular coordinates of the points given in spherical coordinates:

(a) (2,30°,60°). (b) (3,150°,30°). (c) (6,90°,120°).

2. Find spherical coordinates of the points given in rectangular coordinates:

(a) (3,0,0). (b) (0,3,0). (c) (1,2,6).

Transform to equations in spherical coordinates the equations numbered 3 to 8:

3. $x^2 + y^2 + z^2 = a^2$.

4. $x^2 + y^2 + z^2 = ax$.

5. $y = 3$.

6. $(x^2 + y^2 + z^2)(x^2 + y^2) = a^4$.

7. $x^2 + y^2 - z^2 = a^2$.

8. $x^2 + y^2 = 2az$.

Describe and sketch the surfaces represented by the equations in spherical coordinates numbered 9 to 16:

9. $\rho = 1$.

10. $\theta = 150°$.

11. $\phi = 90°$.

12. $\rho \sin \phi \cos \theta = 3$.

13. $\rho = a \sin \phi \sin \theta$.

14. $\rho \sin \phi = 1$.

15. $\rho = \cos \theta$.

16. $\rho = a \sin \phi \cos \theta$.

17. Find rectangular coordinates of the points given in polar coordinates:

(a) (1,0°,90°,90°). (b) (1,90°,0°,90°). (c) (2,54.7°,54.7°,54.7°).

Transform the equations numbered 18 to 23 to equations in polar coordinates:

18. $x^2 + y^2 + z^2 = a^2$.

19. $x^2 + y^2 = z^2$.

20. $x + y + z = 5$.

21. $x^2 + y^2 - z^2 = a^2$.

22. $x^2 - y^2 - z^2 = a^2$.

23. $(x^2 + y^2 + z^2)^2 = a^2(x^2 + y^2)$.

Sketch the surfaces represented by polar equations numbered 24 to 29:

24. $\alpha = 60°$.

25. $\alpha = 90°$.

26. $\beta = 90°$.

27. $\rho \cos \gamma = 3$.

28. $\rho^2(1 - 2 \cos^2 \gamma) = a^2$.

29. $\rho^2(1 - 2 \cos^2 \alpha) = a^2$.

APPENDIX I

Reference Material

1. Solution of equations

The quadratic equation

$$Ax^2 + Bx + C = 0 \tag{1}$$

may be solved for x by using the formula:

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{2}$$

To solve an equation having the form (1), replace A, B, and C in (2) by the corresponding values taken from the given equation.

A method of solving more general equations in x consists in transferring all terms to the same side of the equals sign, factoring the result, equating each factor to zero, and solving the resulting equations. For example, to solve

$$x(x^2 - 4)(x^3 - 1) = 0, \tag{3}$$

factor the left member, equate each factor to zero to get

$$x = 0,\ x - 2 = 0,\ x + 2 = 0,\ x - 1 = 0,\ x^2 + x + 1 = 0, \tag{4}$$

and solve the resulting equations for x to obtain

$$x = 0,\ x = 2,\ x = -2,\ x = 1,$$
$$x = (-1 \pm \sqrt{1^2 - 4})/2 = -\tfrac{1}{2} \pm \sqrt{-3}/2.$$

2. Logarithms

If a is a positive number different from 1, then

$$\log_a n = L, \text{ if } a^L = n. \tag{5}$$

From Equation (5) we see that

$$\log_a 1 = 0. \tag{6}$$

The following laws of logarithms are used frequently:

$$\log_a (mn) = \log_a m + \log_a n. \tag{7}$$

$$\log_a (m/n) = \log_a m - \log_a n. \tag{8}$$

$$\log_a (m)^n = n \log_a m. \tag{9}$$

3. Determinants

A *determinant* is a square array of quantities symbolizing the algebraic sum of certain products of the quantities. The meanings of second-order and third-order determinants are given by the following expressions:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1. \tag{10}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1). \tag{11}$$

Determinants of higher order than the third can be expressed in terms of determinants of lower order by the plan suggested in the development of (11). They are used to give neat, concise expressions for various quantities and equations arising in Analytic Geometry.

4. Angles. Degrees and radians

An angle is generated by turning a line about a point. Thus, in Figure 1, angle AOB may be thought of as generated by rotating line OA about O. When the direction of turning is counterclockwise, the angle generated is positive; when the direction of turning is clockwise, the angle generated is negative.

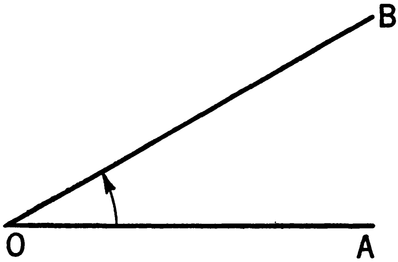

FIG. 1.

Angles are generally measured in degrees, minutes, and seconds or in radians.

$$\begin{aligned} &1 \text{ complete revolution} = 360° \\ &1 \text{ degree} = 60 \text{ minutes} \\ &1 \text{ minute} = 60 \text{ seconds} \end{aligned} \tag{12}$$

A *radian* is the angle (see Figure 2) subtended at the center of a circle by an arc equal in length to a radius. There are 2π radians in 360°. Hence,

$$\pi \text{ radians} = 180°$$

$$1 \text{ radian} = (180/\pi)° = 57°\ 17'\ 44'' \quad (13)$$

$$1° = (\pi/180) \text{ radian} = 0.017453 \text{ radian.}$$

The arc s subtending an angle of θ radians at the center of a circle of radius a (see Figure 2) is given by

$$s = r\theta. \quad (14)$$

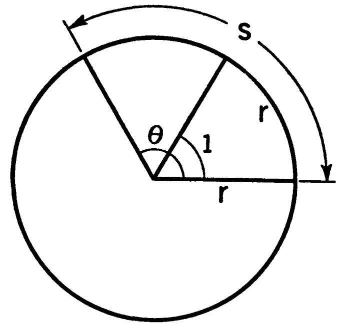

FIG. 2.

5. Definitions of trigonometric functions

If in a right triangle (see Figure 3) the side opposite the acute angle A is called *op*, the side adjacent to A, *adj*, and the hypotenuse, *hyp*, then the trigonometric functions of angle A are defined by

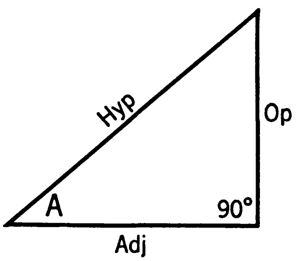

FIG. 3.

$$\sin\theta = \frac{\text{op}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{op}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad (15)$$

$$\tan\theta = \frac{\text{op}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{op}}$$

More generally, if θ is the angle that a line from the origin of a rectangular coordinate system to a point (x,y) in a plane makes with the x-axis, and $r = \sqrt{x^2 + y^2}$ (see Figure 4), then the trigonometric functions of θ are defined by

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x} \qquad (16)$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

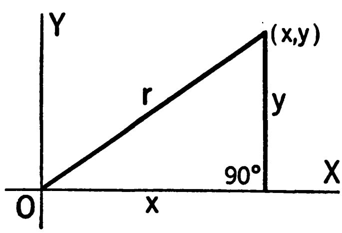

FIG. 4.

The values of the sine, cosine, and tangent of certain angles are contained in the following table:

Angle	$0°$	$90°$	$180°$	$270°$	$30°$	$60°$	$45°$
Sine	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$
Cosine	1	0	-1	0	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$
Tangent	0	None	0	None	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	1

6. Relations among the trigonometric functions

Simple identities.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \sec\theta = \frac{1}{\cos\theta},$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}. \tag{17}$$

$$\sin^2\theta + \cos^2\theta = 1. \tag{18}$$

$$1 + \tan^2\theta = \sec^2\theta. \tag{19}$$

$$1 + \cot^2\theta = \csc^2\theta. \tag{20}$$

Reduction formulas.

$$\sin(-\theta) = -\sin\theta,\ \cos(-\theta) = \cos\theta,\ \tan(-\theta) = -\tan\theta. \tag{21}$$

$$\sin(90° \pm \theta) = \cos\theta,\ \sin(180° \pm \theta) = \mp\sin\theta. \tag{22}$$

$$\cos(90° \pm \theta) = \mp\sin\theta,\ \cos(180° \pm \theta) = -\cos\theta. \tag{23}$$

$$\tan(90° \pm \theta) = \mp\cot\theta,\ \tan(180° \pm \theta) = \pm\tan\theta. \tag{24}$$

Addition formulas.

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi. \tag{25}$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi. \tag{26}$$

$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}. \tag{27}$$

Double-angle formulas.

$$\sin 2\theta = 2\sin\theta\cos\theta. \tag{28}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta. \tag{29}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}. \tag{30}$$

Half-angle formulas.

$$\sin \tfrac{1}{2}\theta = \pm\sqrt{(1 - \cos \theta)/2}. \tag{31}$$

$$\cos \tfrac{1}{2}\theta = \pm\sqrt{(1 + \cos \theta)/2}. \tag{32}$$

$$\tan \tfrac{1}{2}\theta = \pm \frac{\sqrt{1 - \cos \theta}}{\sqrt{1 + \cos \theta}}. \tag{33}$$

7. The law of sines and the law of cosines

If the parts of a triangle are designated by letters as shown in Figure 5, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}. \tag{34}$$

$$a^2 = b^2 + c^2 - 2bc \cos A. \tag{35}$$

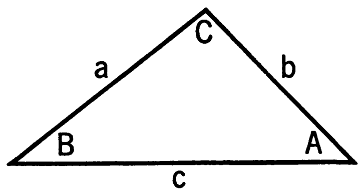

FIG. 5.

APPENDIX II

Table of Miscellaneous Functions

n	$\sqrt{n}$	$\log_{10} n$	$\log_e n$	$e^{\frac{n}{10}}$	$e^{-\frac{n}{10}}$	$\sin n°$	$\cos n°$	$\tan n°$	$n°$ to radians
0	.00	—	—	1.00	1.000	.000	1.000	.000	.000
1	1.00	.000	.000	1.11	.905	.017	1.000	.017	.017
2	1.41	.301	.693	1.22	.819	.035	.999	.035	.035
3	1.73	.477	1.099	1.35	.741	.052	.999	.052	.052
4	2.00	.602	1.386	1.49	.670	.070	.998	.070	.070
5	2.24	.699	1.609	1.65	.607	.087	.996	.087	.087
6	2.45	.778	1.792	1.82	.549	.105	.995	.105	.105
7	2.65	.845	1.946	2.01	.497	.122	.993	.123	.122
8	2.83	.903	2.079	2.23	.449	.139	.990	.141	.140
9	3.00	.954	2.197	2.46	.407	.156	.988	.158	.157
10	3.16	1.000	2.303	2.72	.368	.174	.985	.176	.175
11	3.32	1.041	2.398	3.00	.333	.191	.982	.194	.192
12	3.46	1.079	2.485	3.32	.301	.208	.978	.213	.209
13	3.61	1.114	2.565	3.67	.273	.225	.974	.231	.227
14	3.74	1.146	2.639	4.06	.247	.242	.970	.249	.244
15	3.87	1.176	2.708	4.48	.223	.259	.966	.268	.262
16	4.00	1.204	2.773	4.95	.202	.276	.961	.287	.279
17	4.12	1.230	2.833	5.47	.183	.292	.956	.306	.297
18	4.24	1.255	2.890	6.05	.165	.309	.951	.325	.314
19	4.36	1.279	2.944	6.69	.150	.326	.946	.344	.332
20	4.47	1.301	2.996	7.39	.135	.342	.940	.364	.349
21	4.58	1.322	3.045	8.17	.122	.358	.934	.384	.367
22	4.69	1.342	3.091	9.03	.111	.375	.927	.404	.384
23	4.80	1.362	3.135	9.97	.100	.391	.921	.424	.401
24	4.90	1.380	3.178	11.02	.091	.407	.914	.445	.419
25	5.00	1.398	3.219	12.18	.082	.423	.906	.466	.436
26	5.10	1.415	3.258	13.46	.074	.438	.899	.488	.454
27	5.20	1.431	3.296	14.88	.067	.454	.891	.510	.471
28	5.29	1.447	3.332	16.44	.061	.469	.883	.532	.489
29	5.39	1.462	3.367	18.17	.055	.485	.875	.554	.506
30	5.48	1.477	3.401	20.09	.050	.500	.866	.577	.524
31	5.57	1.491	3.434	22.20	.045	.515	.857	.601	.541
32	5.66	1.505	3.466	24.53	.041	.530	.848	.625	.559
33	5.74	1.519	3.497	27.11	.037	.545	.839	.649	.576
34	5.83	1.531	3.526	29.96	.033	.559	.829	.675	.593
35	5.92	1.544	3.555	33.12	.030	.574	.819	.700	.611
36	6.00	1.556	3.584	36.60	.027	.588	.809	.727	.628
37	6.08	1.568	3.611	40.45	.025	.602	.799	.754	.646
38	6.16	1.580	3.638	44.70	.022	.616	.788	.781	.663
39	6.24	1.591	3.664	49.40	.020	.629	.777	.810	.681
40	6.32	1.602	3.689	54.60	.018	.643	.766	.839	.698
41	6.40	1.613	3.714	60.34	.017	.656	.755	.869	.716
42	6.48	1.623	3.738	66.69	.015	.669	.743	.900	.733
43	6.56	1.633	3.761	73.70	.014	.682	.731	.933	.750
44	6.63	1.643	3.784	81.45	.012	.695	.719	.966	.768
45	6.71	1.653	3.807	90.02	.011	.707	.707	1.000	.785
46	6.78	1.663	3.829	99.48	.010	.719	.695	1.036	.803
47	6.86	1.672	3.850			.731	.682	1.072	.820
48	6.93	1.681	3.871			.743	.669	1.111	.838
49	7.00	1.690	3.892			.755	.656	1.150	.855

TABLE OF MISCELLANEOUS FUNCTIONS (*Continued*)

n	$\sqrt{n}$	$\log_{10} n$	$\log_e n$	$\sin n°$	$\cos n°$	$\tan n°$	$n°$ to radians
50	7.07	1.699	3.912	.766	.643	1.192	.873
51	7.14	1.708	3.932	.777	.629	1.235	.890
52	7.21	1.716	3.951	.788	.616	1.280	.908
53	7.28	1.724	3.970	.799	.602	1.327	.925
54	7.35	1.732	3.989	.809	.588	1.376	.942
55	7.42	1.740	4.007	.819	.574	1.428	.960
56	7.48	1.748	4.025	.829	.559	1.483	.977
57	7.55	1.756	4.043	.839	.545	1.540	.995
58	7.62	1.763	4.060	.848	.530	1.600	1.012
59	7.68	1.771	4.078	.857	.515	1.664	1.030
60	7.75	1.778	4.094	.866	.500	1.732	1.047
61	7.81	1.785	4.111	.875	.485	1.804	1.065
62	7.87	1.792	4.127	.883	.469	1.881	1.082
63	7.94	1.799	4.143	.891	.454	1.963	1.100
64	8.00	1.806	4.159	.899	.438	2.050	1.117
65	8.06	1.813	4.174	.906	.423	2.145	1.134
66	8.12	1.820	4.190	.914	.407	2.246	1.152
67	8.19	1.826	4.205	.921	.391	2.356	1.169
68	8.25	1.833	4.220	.927	.375	2.475	1.187
69	8.31	1.839	4.234	.934	.358	2.605	1.204
70	8.37	1.845	4.248	.940	.342	2.747	1.222
71	8.43	1.851	4.263	.946	.326	2.904	1.239
72	8.49	1.857	4.277	.951	.309	3.078	1.257
73	8.54	1.863	4.290	.956	.292	3.271	1.274
74	8.60	1.869	4.304	.961	.276	3.487	1.292
75	8.66	1.875	4.317	.966	.259	3.732	1.309
76	8.72	1.881	4.331	.970	.242	4.011	1.326
77	8.77	1.886	4.344	.974	.225	4.331	1.344
78	8.83	1.892	4.357	.978	.208	4.705	1.361
79	8.89	1.898	4.369	.982	.191	5.145	1.379
80	8.94	1.903	4.382	.985	.174	5.671	1.396
81	9.00	1.908	4.394	.988	.156	6.314	1.414
82	9.06	1.914	4.407	.990	.139	7.115	1.431
83	9.11	1.919	4.419	.993	.122	8.144	1.449
84	9.17	1.924	4.431	.995	.105	9.514	1.466
85	9 22	1.929	4.443	.996	.087	11.430	1.484
86	9.27	1.934	4.454	.998	.070	14.301	1.501
87	9.33	1.940	4.466	.999	.052	19.081	1.518
88	9.38	1.944	4.477	.999	.035	28.636	1.536
89	9.43	1.949	4.489	1.000	.017	57.290	1.553
90	9.49	1.954	4.500	1.000	.000		1.571
91	9.54	1.959	4.511				
92	9.59	1.964	4.522				
93	9.64	1.968	4.533				
94	9.70	1.973	4.543				
95	9.75	1.978	4.554				
96	9.80	1.982	4.564				
97	9.85	1.987	4.575				
98	9.90	1.991	4.585				
99	9.95	1.996	4.595				
100	10.00	2.000	4.605				

Answers to Exercises

§2, page 5

1. $6\sqrt{2}$ mi. northeast. **2.** 5 at $53° 8'$ with the 3-unit vector. **3.** 14.5 mi. northeast. **5.** 6 units south. **7.** Zero. **8.** (a) $x - 27$. (c) 52 northeast. **9.** $\underset{\rightarrow}{2}$. **11.** $20\sqrt{2}$ southwest.

§3, page 7

2. (a) 3. (b) 5. (c) 10. (d) $|n - m|$. (e) $|m + n|$. **3.** (a) 3.6. (b) -1.6. **5.** A vector, mag. 2 units, direction Pos. **7.** (a) 1,3. (c) 4,1. (e) $\pm\frac{7}{2}$.

§4, page 10

2. On a line parallel to the y-axis and: (a) 3 units to the right of it. (b) 2 units to the right of it. (c) 3 units to the left of it. (d) Coinciding with the y-axis. **3.** On a line parallel to the x-axis and: (a) 2 units below it. (b) 3 units above it. (c) Coinciding with it. **5.** (a) (8,3), (5,3). (c) (2,9), (2,6). **6.** (a) $(3,-1)$, $(0,-1)$. (c) $(-3,5)$, $(-3,2)$. **7.** (a) (1,1), (1,5). (c) $(\pm 1,-4)$. **8.** (a) (2,2), (3,3). (c) $(5,-6)$, $(7,-8)$. **9.** (a) (15,9). (c) (1,5). **10.** (a) $(-2,-3)$. (c) $(10,-1)$. **11.** 23. **12.** (a) $\frac{13}{2}$. (c) 24.

§6, page 13

1. 9, 14, $a^2 + 5$, 6. **2.** 0, -6, $h^4 + h^2$. **3.** 13, 26, 0. **5.** 5. **7.** $2y^3 - 3y$. **9.** 108.

§8, page 18

2. (a) No. (b) No. (c) No. (d) Yes. (e) Yes. (f) Yes. (g) No. (h) No. **10.** (a) 20. (c) $\frac{24}{7}$. **14.** (a) Yes. (c) Yes. (e) No. Yes.

§9, page 22

1. (a) X: ± 5; Y: ± 5. (c) X: ± 4. (e) X: $\pm 1, 4$; Y: ± 2. **2.** (a) x-axis, y-axis, origin. (c) y-axis. (e) None. (g) y-axis. (i) y-axis. **3.** (a) $x < -4$. (c) $|y| < 2$. (e) $|y| < 1$. (g) $x < -2$.

§10, page 25

1. $x'^2 - y'^2 = 4$. **2.** $y'^2 = 4x'$. **3.** $x'^2 + 4y'^2 = 4$.
5. $y' = x'^3 + x'$. **14.** $x - 2 = 0, y + 3 = 0, (2,-3)$.
15. $x + 3 = 0, y = 0, (-3,0)$. **17.** $(-1,0)$.
19. $x + 1 = 0, y - 3 = 0, (-1,3)$.

§12, page 30

1. $(9,-3)$. **2.** $(1,2)$. **3.** $(4,4), (-4,-4)$.
5. $(3,4), (-3,4), (3,-4), (-3,-4)$. **7.** $(0,0), (2,6), (-2,-6)$.
10. $m = 3, n = -4$. **11.** $k = 5, (1,1)$.
13. $(\frac{2}{3},-\frac{2}{3}), (3,4), (-1.4,-4.8)$. **14.** (a) $8\sqrt{2}$. (c) $\sqrt{35}$.
15. $(2,2), (2,-2), (-2,2), (-2,-2), (\frac{2}{7}\sqrt{105},\frac{2}{21}\sqrt{105})$,
$(-\frac{2}{7}\sqrt{105},-\frac{2}{21}\sqrt{105}), (\frac{4}{11}\sqrt{55},-\frac{2}{11}\sqrt{55}), (-\frac{4}{11}\sqrt{55},\frac{2}{11}\sqrt{55})$.
20. Two. **21.** $6(1 + \sqrt{13})$.

§14, page 34

1. $y = 2x + 1$. **2.** $2x = y^2$. **3.** $x^2 + y^2 = 25$. **5.** $y = \pm 2$.
7. $x = \pm y$. **9.** 8. **11.** $(4,6), (-24,34)$. **13.** $2x = a + b$.

§15, page 39

1. (a) 1, 3. (c) 7, -5. **2.** (a) 5. (c) 3. **7.** (a) $(6,4)$. (b) (a,b).
9. (a) $(0,\pm h)$. (b) $(\frac{1}{2}b,\pm h)$. **11.** (a) $(3,\pm 3), (6,0)$.
(b) $(s/\sqrt{2},\pm s/\sqrt{2}), (\sqrt{2}s,0)$. **19.** (a) $(x - 3)^2 + (y + 2)^2 = 25$.
(c) $x + y = 3$. (e) $3x^2 + 3y^2 - 24x - 40y + 136 = 0$.
21. $(3,4), (-1.4,-4.8)$. **24.** 7. **25.** 4. **27.** -10. **29.** 9, -7.
31. $6\sqrt{5}$. **33.** $\frac{4}{3}\sqrt{5}$. **35.** (a) $\frac{7}{2}$. (b) 15.

§17, page 45

1. (a) 1, 45°. (b) $\frac{3}{5}$, 31.0°. (c) $-\sqrt{3}$, 120°. (d) 2, 63.4°.
(e) 0, 0°. **2.** 0, none. **3.** (a) $\frac{2}{11}$. (b) $-\frac{1}{5}$. (c) 0. (d) None.
5. $-1, -\frac{4}{3}, -\frac{8}{7}, 1, \frac{3}{4}, \frac{7}{8}$.
6. *AD* parallel to *BC*, *AC* parallel to *BD*, *AC* perpendicular to *AD*, *AC* perpendicular to *BC*, *BD* perpendicular to *AD*,
BD perpendicular to *BC*. **7.** (a) 32.1°. (b) 90°. **9.** 56.3°, 33.7°, 90°.
11. (a) 72.3°, 107.7°. (b) 71.6°, 108.4°. **13.** $(3,-10), (-3,11)$.
14. $3x + 5y = 45$. **15.** $2x - y = 1$. **17.** $x + 2y = 8$. **19.** $-\frac{13}{3}$.
21. 1. **26.** $-7 - 5\sqrt{2}$.

§18, page 48

1. (a) 90°. (b) 0°. **3.** (a) 270°. (b) 180°. **5.** (a) 120°. (b) 30°. **7.** −315°. **9.** −225°. **11.** 540°. **13.** 558.4°. **15.** 90°, −270°. **16.** −2033.1°.

§19, page 50

1. (0,4), $(-\frac{7}{2},5)$, $(-\frac{1}{2},7)$. **2.** (7,0), (9,−6). **3.** (a) (3,0), $(0,\frac{9}{2})$. (b) (2,0), (0,3). (c) (12,0), (0,18). **5.** (−3,6), (−11,14). **7.** (a) $(3,\frac{1}{2})$. (b) (−3.6,43.2). **9.** (a) (1,−2). (b) $(\frac{8}{3},\frac{8}{3})$.

§20, page 53

1. $4x - 2y = -13$. **2.** $x^2 + y^2 - 20y + 84 = 0$.
3. $x + 3y = 10$. **5.** $y^2 = 8x - 16$. **7.** $y^2 = 4x + 6y + 3$.
9. $y^2 = -20x + 6y + 51$. **11.** $x^2 + y^2 = 6x$. **13.** $3x - y = 8$.
15. $6x - 5y = 0$. **17.** $x - 5y + 10 = 0$. **19.** $x - y + 5 = 0$.
21. $x + 3y = 8$. **23.** $(x - a)^2 + y^2 = a^2$.
25. $(x - 3)^2 + (y - 4)^2 = 16$. **27.** $(-\frac{4}{3},0)$. **29.** $(2,-1)$.
31. (1,0), (13,24). **33.** $2xy = 9y - 5x$.
34. $y^2 - 12x + 12 = 0$ if $x \geqq 2$, $y^2 + 4x - 20 = 0$ if $x \leqq 2$.
35. $x^2 = 4y + 8$ if $y \geqq 1$, $x^2 = 24 - 12y$ if $y \leqq 1$.
36. $x^2 + 6y - 3 = 0$.*
37. $y^2 = 9 - 6x$ if $x \geqq -3$, $y^2 = 18x + 81$ if $x \leqq -3$.

§21, page 57

1. $2x - y = 2$. **3.** $x + y = 1$. **5.** $x = 3$. **7.** $5x + 4y = 22$.
9. $(y - y_1)/(x - x_1) = (y_2 - y_1)/(x_2 - x_1)$. **11.** $y = mx + b$.
13. $y = mx - ma$. **15.** $\sqrt{3}x + y + 6 = 0$. **17.** $4x + 5y = 41$.
18. $9x + y = 41$.
19. $(3 - 4\sqrt{3})x - (4 + 3\sqrt{3})y + 14\sqrt{3} - 23 = 0$.

§22, page 60

2. $y = 3x + 6$. **3.** $4x - y = 11$. **5.** $5x - 3y = 15$. **7.** $x = 5$.
9. $y = 0$. **10.** (a) $-\frac{3}{2}$, 6. (c) $\frac{2}{3}$, $-\frac{7}{3}$. **11.** (a) 6, 2.
13. $4x + 3y + 20 = 0$.
15. $3x - 2y = 0$, $3x + 2y = 12$, $3x - 10y + 24 = 0$.
17. $2x - 3y = 24$. **18.** $x + 2y + 1 = 0$. **19.** $7x + 5y = 70$.

* The locus is given by $\sqrt{x^2 + (y + 1)^2} = |y - 1| + 1$. For $y > 1$, we have $x^2 + (y + 1)^2 = (y - 1 + 1)^2$, or $x^2 = -2y - 1$; but this contains no points with $y > 1$. Hence, the only solution is the one given.

§23, page 62

1. (a) 2. (c) $\frac{1}{2}$. (e) 0.
2. I and II are parallel, III is perpendicular to I and II.
3. (a) $2x + y = 4$. (c) $y = -6$. **4.** (a) $3x - y + 21 = 0$.
(c) $x + 6 = 0$. **7.** $x - y = 3$. **9.** $(\frac{7}{2},-2)$. **13.** 0. **15.** 78.7°.
16. $2x + y = 2$, $x - 2y = 6$. **17.** $x - 2y = \pm 6$. **18.** $-\frac{1}{2}$, -2.
19. $4x - 5y = 41$. **21.** $x + 3y = 24$. **23.** $(-3,4)$, 10.
25. (a) 3. (c) -1. **26.** (a) Coincident. (b) Perpendicular.
(c) Parallel. **27.** $(-4,-28)$, $4x - y = 12$.

§24, page 67

1. (a) $\frac{5}{13}x - \frac{12}{13}y = 3$. (c) $\frac{3}{5}x + \frac{4}{5}y = 0$. (e) $y = \frac{3}{2}$. (g) $-x = \frac{4}{3}$.
3. (a) 7. (c) 0. (e) 3. **5.** 64/5. **6.** (a) $x + y = 10\sqrt{2}$.
(c) $-x - y = 10\sqrt{2}$. (e) $y = 4$. **7.** (a) 5, 60°. (c) $\frac{5}{4}$, 0°.
9. (a) $\frac{4}{5}$. (c) $21/\sqrt{10}$. **10.** $4\sqrt{5}$, $26/\sqrt{5}$, $6/\sqrt{5}$. **11.** $K = \pm\frac{3}{4}$.
12. $x + y = 4$, $x - 7y + 20 = 0$.
13. (a) $4x - 3y = 25$, $3x + 4y = 25$.
(b) $2x - 3y = 13$, $3x + 2y = 13$.

§25, page 71

1. (a) -4. (c) $-5\sqrt{2}$. (e) $-2\sqrt{10}$. **2.** $52/\sqrt{58}$, 26.
3. $\frac{53}{2}$, $53/\sqrt{65}$. **7.** $47/\sqrt{61}$. **9.** $3x + 4y = \pm 50$. **11.** 16.
12. $(2,-\frac{107}{12})$, $(2,-\frac{1}{4})$. **13.** $2x + 4y = 35$.
14. $3x + 4y = -55$, $3x + 4y = 45$, $3x + 4y = -5$. **15.** $\frac{35}{4}$.
16. 3, $\frac{5}{3}$. **17.** $7x - 23y = 34$, $23x + 7y = 136$. **19.** $(1,-\frac{10}{3})$.
21. (4,4), 4. **22.** (1,1), $\frac{2}{5}$.
23. $3x + 4y + 12 = 0$, $3x + 4y - 48 = 0$.
25. $x^2 + 4xy + y^2 - 16x - 16y + 32 = 0$.
27. $(9x - 13y - 80)(3x - y - 20) = 0$.

§26, page 73

1. $y = mx$. **2.** $y - 2 = m(x + 3)$. **3.** $x\cos\omega + y\sin\omega = 10$.
5. $2x - 3y = k$. **7.** $x - \sqrt{3}y = k$. **9.** $(x/a) + ay/20 = 1$, $a > 0$.
10. $4x - 3y = k$. **11.** Slope $= -\frac{5}{6}$. **13.** x-intercept 5.
14. Algebraic sum of intercepts = 1. **15.** Distant 20 from (0,0).
17. (11) 4. (13) -3. **18.** 5, -1. **19.** $x\cos\omega + y\sin\omega = 3$; 3, $\frac{15}{4}$.
23. -3. **24.** -2. **25.** -1, $-\frac{5}{4}$. **27.** $-\frac{10}{17}$. **29.** $0.4 \pm 0.1\sqrt{166}$.
30. $x + y = a$, $y = mx$.

§27, page 76

1. (a) $x^2 + y^2 = 9$. (c) $(x + 5)^2 + y^2 = 25$.
3. $(x - 2)^2 + (y - 1)^2 = 5$. **5.** None. **6.** $x^2 \pm 10x + y^2 = 0$.
7. $x^2 + (y \pm 4)^2 = 25$. **9.** $(x - 2)^2 + (y - 3)^2 = 4$.
11. $(-4,6)$, $(3,-1)$. **12.** (a) $(x - 4)^2 + (y + 3)^2 = 2$.
(c) $(x + 1)^2 + (y + 3)^2 = 10$. **13.** (a) Center on y-axis and radius 4.
(c) Center on $x = y$, radius 6. (e) Radius 5.
(f) Center on $x = y$, tangent to both axes.
17. $x'^2 + y'^2 = a^2 + b^2 + c$.

§28, page 78

1. $(1,4)$, 5. **3.** $(2,-3)$, $\frac{5}{2}$. **5.** $(0,\frac{7}{2})$, $\frac{7}{2}$.
7. $(x + 1)^2 + (y - 2)^2 = 37$. **9.** (d). **10.** $a = 2$, $c^2 + d^2 > 8e$.
11. $(-2,0)$, 4. **13.** $x^2 + y^2 - ax - by = 0$.
14. $(x - 4)^2 + (y + 2)^2 = 25$. **15.** Yes. **16.** (a) $x^2 + y^2 = 4x$.
17. $2x^2 + 2y^2 - 2ax - 2by = c^2 - (a^2 + b^2)$.
19. (a) $x^2 + y^2 - 2a\sqrt{3}y = a^2, y > 0$; $x^2 + y^2 + 2a\sqrt{3}y = a^2, y < 0$.
(b) $x^2 + y^2 + 2ay = a^2, y > 0$; $x^2 + y^2 - 2ay = a^2, y < 0$.
21. $(0,-1)$, 2. **22.** $(\frac{5}{2},-\frac{1}{2})$, $\sqrt{3}$. **23.** $x^2 + y^2 = l^2$.

§29, page 81

1. $(1 - h)^2 + (-2 - k)^2 = a^2$. **2.** $h - 2k = 3$. **3.** $k = \pm a$.
4. $h = \pm a, k = 4$. **5.** $h^2 + k^2 = a^2, h = 2$.
7. $\dfrac{3h + 4k - 5}{\pm 5} = a$, $(1 - h)^2 + (\frac{1}{2} - k)^2 = a^2$.
8. $h^2 + k^2 = a^2$, $(2 - h)^2 + (-1 - k)^2 = a^2$, $h + k = 1$.
9. $(1-h)^2 + (1-k)^2 = a^2$, $(2-h)^2 + (2-k)^2 = a^2$, $h^2 + (4-k)^2 = a^2$.
11. $(x - 2)^2 + (y - 3)^2 = 4$. **13.** $(x - 2)^2 + (y - 3)^2 = \frac{225}{2}$.
15. $(x - 2)^2 + (y - 1)^2 = 1$, $(x - 4)^2 + (y - 2)^2 = 4$.
17. $(x - 4)^2 + (y - 4)^2 = 18$, $(x + 2)^2 + (y + 2)^2 = 18$.
19. $(x-\sqrt{8})^2 + y^2-2(4-\sqrt{8})y = 0$, $(x+\sqrt{8})^2 + y^2-2(4+\sqrt{8})y = 0$.
21. $(x - 3)^2 + (y - 1)^2 = 1$, $(x + 1)^2 + (y - 5)^2 = 25$.
23. $(x - 1)^2 + (y + 2)^2 = 5$. **25.** $(x + 2)^2 + (y - 1)^2 = 13$.
29. $x^2 + y^2 - 6x - 4y = 0$. **30.** $(x - 4)^2 + (y - 3)^2 = 25$.

§30, page 85

2. (a) 0. (b) -1.4. **3.** $33.7°$.
4. $x^2 + y^2 - 4x - 5 + k(x^2 + y^2 - 5) = 0$, $x = 0$.

5. (a) $x^2 + y^2 - 40x + 32y = 0$.
(b) $13x^2 + 13y^2 + 24x - 128y + 204 = 0$. **8.** 5. **9.** 12.
14. The system of circles tangent to $ax + by = c$ at (m,n), except $v = 0$.
15. The system concentric with $u = 0$ or $v = 0$, except $v = 0$.

§32, page 88

2. P_1: $(-6, 210°)$, $(-6, 150°)$; P_3: $(-12\sqrt{3}, 60°)$, $(12\sqrt{3}, -120°)$; P_5: $(-5, 120°)$, $(5, -60°)$; P_7: $(-6, 270°)$, $(-6, -90°)$.
5. (a) Circle, center pole, radius 5. (b) Circle, center pole, radius a.

§33, page 91

1. (a) $(5,5)$. (c) $(-3\sqrt{3},-3)$. (e) $(0,-3)$. (g) $(0,3)$. (i) $(-3,0)$.
2. (a) $(3\sqrt{2}, 45°)$. (c) $(10, 233.1°)$. (e) $(6, 0°)$. (g) $(8, 90°)$.
(i) $(0, \theta)$. **3.** $\rho \cos\theta = a$. **5.** $\rho = a$. **7.** $\rho - 6\cos\theta = 0$.
9. $\rho = 6\cos\theta + 8\sin\theta$. **11.** $\rho^2(\cos^2\theta + 4\sin^2\theta) = 4$.
13. $\rho^2 = \cos^2\theta - \sin^2\theta = \cos 2\theta$. **15.** $x^2 + y^2 = a^2$. **17.** $y = b$.
19. $x^2 + y^2 = ax$. **21.** $ax + by = c$. **23.** $(x^2 + y^2)^3 = 4x^2y^2$.
25. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

§36, page 98

2. $\theta = \pm 45°$. **3.** $\theta = 0$. **4.** 90°. **5.** 180°. **7.** 45°.
9. Symmetric to $\theta = 90°$.
11. Symmetric to pole, $\theta = 90°$, and polar axis.
13. Symmetric to polar axis, $\theta = 90°$, and pole.
15. $90° < \theta < 180°$, $270° < \theta < 360°$. **17.** $210° < \theta < 330°$.

§38, page 102

1. $\rho = a$. **2.** $\rho = 2a\sin\theta$. **3.** $\rho = -2a\cos\theta$.
5. $\rho = -b\csc\theta$. **7.** $\rho = a\cos\theta + b\sin\theta$.
9. $\rho\cos(150° - \theta) = a$. **11.** $\sin\theta = \frac{1}{2}$. **13.** $\rho = a + a\sec(\theta - \alpha)$.
14. $\rho = 2a\cos\theta \pm m$.* **15.** $\rho = \pm 2r\sin^2\theta\sec\theta$.
16. $\rho = \pm(2a\cos\theta - a\sec\theta)$, $\rho^2 = 2a^2(1 - \sin 2\theta)$,
$\rho^2 = 2a^2(2\tan^2\theta - 2\tan\theta + 1)$.

§39, page 106

1. $(\frac{1}{2}, \pm 60°)$. **3.** $(\sqrt{2}, 45°)$, $(\sqrt{2}, 135°)$. **5.** None.
7. $(2\sqrt{3}, 60°)$, $(2\sqrt{3}, 120°)$. **9.** Pole, $(\frac{1}{2}\sqrt{3}, 60°)$, $(-\frac{1}{2}\sqrt{3}, -60°)$.
11. $(\sqrt{2}, \pm 45°)$, pole. **13.** $(\frac{1}{2}\sqrt{2},\ 22.5° + k45°)$, $k = 1, 2, \cdots, 8$; pole.

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* $\rho = 2a\cos\theta + m$ represents the complete locus.

14. $(1,\ 22.5° + k45°)$, $k = 1, 2, \cdots, 8$. **15.** $(\pm 2, 30°)$.
16. $(-1, \pm 90°)$, $(-\frac{1}{2}, \pm 60°)$, $(-0.219, \pm 38.3°)$, pole.
17. $(1,0), (1,180°), (\frac{1}{2}, -30°), (\frac{1}{2}, -150°), (-0.219, 51.3°), (-0.219, 128.7°)$, pole. **18.** $(0.219, \pm 141.3°)$, $(-\frac{1}{2}, \pm 60°)$, $(-1, \pm 90°)$, pole.
19. $(1, \pm 60°)$, $(-1, \pm 60°)$.
20. $(\frac{1}{2}, 60°)$, $(\frac{1}{2}, 300°)$, $(-\frac{1}{2}, -60°)$, $(-\frac{1}{2}, -300°)$.

§41, page 110

7. $x - y = 3$. **9.** $x^2 + y^2 = 1$. **11.** $x = \sqrt{y + 3}$.
13. $\frac{1}{4}(x - 3)^2 + \frac{1}{9}(y - 2)^2 = 1$. **15.** $\sqrt{x} + \sqrt{y} = \sqrt{a}$.
17. $x^2 - y^2 = 4$. **18.** $x = 2\,(1 + t^2)$, $y = 2t/(1 + t^2)$.
19. $x = (3 + t)/(1 - t^2)$, $y = (3t + t^2)/(1 - t^2)$.
21. $x = 3/(1 + t + t^2)$, $y = 3t/(1 + t + t^2)$.
22. $(x - 1)^2 + (y - 2)^2 = 1$, $\rho^2 - 2\rho\cos\theta - 4\rho\sin\theta + 4 = 0$.
23. $3x = 4y$, $\tan\theta = \frac{3}{4}$.
24. $y^2 - (x - 1)^2 = 1$, $\rho^2(\sin^2\theta - \cos^2\theta) + 2\rho\cos\theta - 2 = 0$.
25. $(x^2 + y^2)^3 = 4x^2y^2$, $\rho = \sin 2\theta$. **26.** $\rho = 8t$, $\theta = 4\pi t$.

§44, page 115

1. $x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$.
2. (a) $\sqrt{3}x - y = \frac{2}{3}\pi a$. (b) $\sqrt{3}x + y = \frac{4}{3}\pi a$. (c) $x = \frac{1}{2}\pi a$.
3. $x = a(\theta - \sin\theta)$, $y = -a(1 - \cos\theta)$.
5. $x = a(\theta - \pi - \sin\theta)$, $y = a(-1 - \cos\theta)$.
7. $x = (a - b)\cos\theta + b\cos[(b - a)\theta/b]$,
$y = (a - b)\sin\theta + b\sin[(b - a)\theta/b]$.
9. $x = 2a\tan\theta$, $y = 2a\cos^2\theta$, $y = 8a^3/(x^2 + 4a^2)$.
10. $x = 2a\sin^2\theta$, $y = 2a\tan\theta\sin^2\theta$, $y^2 = x^3/(2a - x)$,
$\rho = 2a\sin\theta\tan\theta$.
11. $x = a(1 \pm \cos\theta)$, $y = a(\sec\theta \pm 1)\sin\theta$. Yes.

§46, page 121

1. $y^2 - 4x + 4 = 0$, $y'^2 = 4x'$. **3.** $x'^2 - y'^2 = 8$.
4. $(1 - e^2)x'^2 + y'^2 = p^2e^2/(1 - e^2)$.

§47, page 124

1. $2y_8 = 16$, $2y_{10} = 8\sqrt{5}$. **2.** $2x_{50} = 40$. **4.** $\frac{100}{9}$.
5. The larger the value of $|p|$, the flatter the curve at its vertex.
7. $x^2 = -6y$, $(0, -\frac{3}{2})$. **8.** $y^2 = -12x$, $x^2 = \frac{3}{2}y$, two. **11.** $(16,16)$.

12. $6\sqrt{5}$. **13.** $x^2 = 1600y$. **15.** (a) $y'^2 = 6x'$. (b) $x'^2 = 4y'$. (c) $y'^2 = 2x'$. (e) $x'^2 = 2py'$. **16.** V: (1,2), (0,0); F: (2,2), (1,0). **17.** V: (0,−2), (0,0); F: (0,0), (0,2). **21.** $(x - 2)^2 = 8(y + 2)$. **22.** $x^2 - 2xy + y^2 + 12x + 4y + 4 = 0$. **23.** $x^2 + 2xy + y^2 + 8x - 12y = 34$, $(\frac{9}{4}, -\frac{5}{4})$.

§48, page 126

5. (a) $x = -\frac{7}{2}$, $(-\frac{5}{2}, 1)$, $(-3, 1)$. (c) $x = \frac{3}{2}$, $(\frac{1}{2}, -2)$, $(1, -2)$. **6.** $(y - 3)^2 = 8x$. **7.** $(x - 5)^2 = 12y$. **10.** $\rho = 8/(1 - \cos\theta)$. **11.** $\rho = 8/(1 + \cos\theta)$. **13.** $\rho = 6/(1 + \cos\theta)$. **22.** $x^2 = -12(y - 3)$, and $x = 0$; a parabola and the y-axis. **23.** $y^2 = 4(x + 1)$, and $y^2 = -8(x - 2)$. **24.** $y^2 = 16(x + 4)$. **25.** $y^2 = 12x + 12$ for $x \geqq 0$, $y^2 = -4x + 12$ for $x \leqq 0$. **27.** $\rho \sin^2\theta = p\cos\theta$.

§50, page 133

1. $\frac{32}{5}$. **3.** 1. **5.** $(0, \pm 4\sqrt{3})$, $y = \pm 16/\sqrt{3}$. **7.** $(0, \pm\sqrt{5})$, $y = \pm 9/\sqrt{5}$. **13.** Nearly round. Long and thin. **14.** 0, No. **15.** Nearly circular.

17. $\dfrac{x^2}{36} + \dfrac{y^2}{9} = 1$. **19.** $\dfrac{x^2}{64} + \dfrac{y^2}{48} = 1$. **20.** $\dfrac{x^2}{6} + \dfrac{y^2}{3} = 1$.

21. $\dfrac{2b^2}{a}$, $2\sqrt{a^2 - b^2}$. **23.** $\frac{5}{2}$, 6.

25. (a) $\frac{1}{25}x^2 + \frac{1}{9}y^2 = 1$. (b) $\frac{1}{16}x^2 + \frac{1}{25}y^2 = 1$. **27.** $36x^2 + 9y^2 = 4a^2$. **29.** $9x^2 + y^2 = a^2$. **30.** (b) $b^2x^2 + 4a^2y^2 = a^2b^2$. **31.** $b^2x^2 + a^2y^2 = a^2b^2$.

§51, page 136

1. $\frac{1}{25}(x - 1)^2 + \frac{1}{9}(y - 3)^2 = 1$, $\frac{1}{9}(x - 1)^2 + \frac{1}{25}(y - 3)^2 = 1$. **3.** $\frac{1}{25}(x - 5)^2 + \frac{1}{9}y^2 = 1$, $\frac{1}{9}(x - 5)^2 + \frac{1}{25}y^2 = 1$. **5.** $(0, \pm\sqrt{5})$, $(1, -2 \pm \sqrt{5})$. **7.** $(0, -3 \pm \sqrt{3})$. **9.** $(\pm\sqrt{3}, -\frac{3}{2})$. **10.** $\frac{1}{25}(x - 2)^2 + \frac{1}{16}(y - 2)^2 = 1$. **11.** $\frac{1}{25}(x - a)^2 + \frac{1}{9}y^2 = 1$. **12.** $\frac{1}{25}x^2 + \frac{1}{16}y^2 = 1$. **13.** $\frac{1}{9}x^2 + \frac{1}{25}y^2 = 1$. **15.** $\frac{1}{16}x^2 + \frac{1}{25}y^2 = 1$. **16.** (a) $(25 + k^2)x^2 + 25(y - k)^2 = 25(25 + k^2)$. (b) $25(x - k)^2 + (25 + k^2)y^2 = 25(25 + k^2)$. **17.** $(x - 2)^2 + 36(y + 2)^2 = 4$. **18.** $\frac{1}{25}(x - 1)^2 + \frac{1}{9}(y + 3)^2 = 1$. **19.** $b^2(x - h)^2 + a^2(y - k)^2 = b^2c^2$. **22.** (a) $\frac{1}{2}$, 10. (c) $|n/m|$, g/n. **23.** 9, $6\sqrt{2}$. **27.** 16, $8\sqrt{3}$. **29.** $\frac{25}{2}$, 10. **32.** $b^2(x - h)^2 + a^2(y - k)^2 = a^2b^2$.

§53, page 145

1. 4.5. **3.** $\frac{32}{3}$. **5.** $\frac{1}{16}x^2 - \frac{1}{9}y^2 = 1$. **7.** $\frac{1}{16}y^2 - \frac{1}{9}x^2 = 1$.
9. $\frac{1}{20}x^2 - \frac{1}{5}y^2 = 1$. **11.** (a) Sharply. (b) Slowly. **12.** Asymptotes.
13. $\frac{1}{2}\sqrt{5}, \sqrt{5}$; 4, 2; $\frac{4}{5}\sqrt{5}, \frac{1}{5}\sqrt{5}$. **15.** $\sqrt{2}, \sqrt{2}$; 4, 4; $\sqrt{2}, \sqrt{2}$.
17. $b^2x^2 - 4a^2y^2 = a^2b^2$. **19.** $\frac{8}{3}\sqrt{17}$. **21.** $\frac{4}{3}\sqrt{3}$.
23. $\frac{1}{16}x^2 - \frac{1}{9}y^2 = 1$. **24.** $(c^2 - a^2)y^2 - a^2x^2 = a^2(c^2 - a^2)$.
25. $x - y = 3$. **27.** $y^2 - 2x^2 = 9$.

§54, page 147

1. $\frac{1}{4}(x-1)^2 - \frac{1}{9}(y-3)^2 = 1$. **3.** $\frac{1}{9}(y+3)^2 - \frac{1}{25}x^2 = 1$.
5. $\sqrt{2}$, (0,1), (−2,1). **7.** 2, $(-2 \pm 2\sqrt{3}, 0)$.
9. $\frac{1}{5}(x-1)^2 - \frac{1}{4}(y-2)^2 = 1$. **11.** $(x-3)^2 - (y-3)^2 = 4$.
13. $\frac{1}{9}(y-k)^2 - \frac{1}{27}x^2 = 1$. **15.** $\frac{1}{9}(y-2)^2 - \frac{1}{7}(x+1)^2 = 1$.
16. $(x-2)^2 - 16(y+2)^2 = 7$. **18.** (a) 3, $\frac{3}{2}$. **19.** 3, $6\sqrt{2}$.
23. 8, $8\sqrt{3}$. **25.** $\frac{15}{2}$, 10. **27.** $b^2(x-h)^2 - a^2(y-k)^2 = a^2b^2$.

§57, page 153

1. $x'^2 - y'^2 = 8$. **3.** $7x'^2 + 4y'^2 = 4$. **5.** $x'^2 - y'^2 = 18$.
7. $x'^2 + 9y'^2 = 9$. **9.** $x'^2 - 4y'^2 = 2y' - x'$.

§59, page 158

1. $x = (4x' - 3y')/5$, $y = (3x' + 4y')/5$.
3. $x = (x' - y')/\sqrt{2}$, $y = (x' + y')/\sqrt{2}$.
5. $x = (\sqrt{2}x' - y')/\sqrt{3}$, $y = (x' + \sqrt{2}y')/\sqrt{3}$.
7. (1) Hyperbola. (2) Ellipse. (3) Parabola. (4) Ellipse.
(5) Parabola. (6) Hyperbola. **8.** $\frac{3}{2}x'^2 + \frac{1}{2}y'^2 + 8y' = 10$.
9. $5x'^2 - 5y'^2 + \sqrt{10}x' + 2\sqrt{10}y' = 350$.
11. $\frac{1}{28}x''^2 + \frac{1}{84}y''^2 = 1$. **13.** $x''^2 = -\frac{3}{5}y''$. **15.** $x'^2 + \frac{1}{4}y'^2 = 1$.
16. $\dfrac{x''^2}{(13/3)} + \dfrac{y''^2}{13} = 1$. **17.** $\dfrac{y'^2}{81} - \dfrac{x'^2}{9} = 1$. **19.** $\dfrac{x''^2}{16} - \dfrac{y''^2}{16} = 1$.
20. $x^2 - 2xy + y^2 + 4\sqrt{2}(x+y) - 8 = 0$, $y''^2 = -4x''$.
21. $7x^2 - 2xy + 7y^2 + 4\sqrt{2}(x+y) - 8 = 0$, $9x''^2 + 12y''^2 = 16$.
23. Yes.

§61, page 163

1. $x = 2, y = 0$. **2.** $x = \pm 1, y = 1$. **3.** $x = 0, y = 0$.
4. $x = \pm 1, y = \pm 2$. **24.** $y = x + 1$. **25.** $y = x - 6$. **27.** $y = x$.

§62, page 169

1. π, 3. **2.** $\frac{2}{3}\pi$, 4. **3.** $\frac{1}{3}\pi$. **4.** $\frac{1}{2}\pi$. **5.** $\frac{2}{3}\pi$. **6.** $\frac{2}{5}\pi$.

§63, page 172

1. (a) $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. (b) 0 to π. (c) Between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. (d) Between 0 and π.

§68, page 182

4. $y = \rho \sin \theta = 1$. 5. $x = \rho \cos \theta = 1$. 6. $y = \rho \sin \theta = \pm\sqrt{2}$. 7. $y = \rho \sin \theta = \pm 1$. 8. $y' = \rho \sin (\theta - 30°) = \frac{3}{4}$.
9. $y' = \rho \sin (\theta - 45°) = \pm\frac{1}{2}a$, $x' = \rho \cos (\theta - 45°) = \pm\frac{1}{2}a$.
10. $\rho \sin (\theta - 60°) = a/\sqrt{3}$, $\rho \sin (\theta + 60°) = -a/\sqrt{3}$.
12. $\rho \sin (\theta - 135°) = \frac{2}{3}\sqrt{2}a$.

§71, page 188

1. $y = 0.93x + 2.2$. 2. $y = 0.00752x + 0.992$.
3. $R = 0.28T + 71$. 4. $w = (1616/I) - 94.8$.
5. $F = 0.1459W + 1.87$. 6. $H = 0.314T + 605$.
7. $l = 4.1(10)^{-5}W + 1200$. 8. $y = 2.0x^2 - 3.1$.
9. $L = 0.00478W + 10.03$. 10. $p = 0.0030v^2 + 0.00$.

§73, page 193

1. $y = 2.07x + 2.9$. 2. $l = 0.050w + 10.1$.
3. $H = 1.26v^3 + 130$. 4. $L = 0.28w + 10.1$. 5. $p = 0.167T + 23.8$

§74, page 197

1. $y = 1.92x^{0.51}$. 2. $y = 290x^{-1.48}$. 3. $y = 480x^{-1.06}$.
4. $R = 0.0603V^{2.02}$. 5. $p = 99v^{-1.30}$. 6. $F = 3.7x^{-2.0}$.
7. $Q = 0.0023v^{1.99}$. 8. $T = 0.00112r^{1.50}$. 9. $D = 2.0h^{0.50}$.

§75, page 200

1. $y = 62(10)^{-0.25x}$. 2. $y = 75(10)^{-0.048x}$. 3. $y = 880(10)^{-0.124x}$.
4. $A = 100(10)^{-0.0115t}$. 5. $N = 61(10)^{0.086t}$.
6. $T = 100(10)^{-0.069t}$. 7. $v = 43(10)^{-0.098t}$.

§76, page 203

1. $y = 2.1 + 0.131x + 0.23x^2$. 2. $y = 0.47 - 0.99x + 0.099x^2$.
3. $y = 111.7 + 1.663x + 0.00437x^2$.

§77, page 204

1. $y = 0.25x + 0.82$. 2. $y = 3.5x^{0.54}$. 3. $y = 9.9(10)^{-0.085x}$.
4. $y = 2.5 + 4.7x + 1.58x^2$. 5. $N = 992(10)^{0.1775t}$.
6. $T = 63.1(10)^{-0.0284t}$. 7. $p = 2.17(10)^{-8}T^{3.85}$.
8. $F = 1000(10)^{-0.217\alpha}$.

§78, page 207

2. (a) (0,0,0). (c) $(a,0,0)$. **3.** (a) $(a,0,0)$. (c) $(0,0,a)$.
4. (a) Right of the yz-plane. (c) On the Pos. side of the xz-plane.
5. (a) In yz-plane. (c) in the xy-plane. **6.** In a plane parallel to the:
(a) yz-plane and 2 units right of it.
(b) xz-plane and 3 units from it on the Pos. side.
(c) xy-plane and three units below it.
(d) yz-plane and 3 units left of it. **7.** A line parallel to the:
(a) z-axis and coinciding with it. (b) z-axis and through (0,1,0).
(c) z-axis and through (1,2,0). (d) y-axis and coinciding with it.
(e) x-axis and through (0,1,2).

§80, page 211

2. C. **3.** A,B. **5.** A,C,D. **7.** -6. **9.** $-\frac{25}{9}$.
24. z-axis, (2,2,0); z-axis, (2,−2,0). **25.** z-axis, (2,3,0).
27. x-axis, (0,4,8); x-axis, (0,−4,8).
29. y-axis, $(\sqrt{3},0,3)$; y-axis, $(-\sqrt{3},0,3)$.
31. y-axis, (2,0,1); y-axis, (1,0,2).

§81, page 214

1. $X = 3$, $Y = 6$, $Z = -2$; $2x + y = 6$, $z = 0$;
$2x - 3z = 6$, $y = 0$; $y - 3z = 6$, $x = 0$.
2. $X = \pm 5$, $Y = \pm 5$, $Z = \pm 5$; $x^2 + y^2 = 25$, $z = 0$;
$x^2 + z^2 = 25$, $y = 0$; $y^2 + z^2 = 25$, $x = 0$;
Planes of symmetry, all three coordinate planes.
3. $X = \pm 8$, $Y = \pm 4$; $x^2 + 4y^2 = 64$, $z = 0$; $x^2 - z^2 = 64$, $y = 0$;
$4y^2 - z^2 = 64$, $x = 0$; Planes of symmetry, all coordinate planes.
5. $Z = \frac{5}{2}$; $x = 0$, $y^2 = 2z - 5$; $y = 0$, $z = \frac{5}{2}$; symmetric to xz-plane.

§83, page 221

1. Sphere. Trace on $x = 3$ a circle of radius 4 and,
on $x = -4$, a circle of radius 3.
3. Hyperboloid of one sheet. Trace on $z = 0$ an ellipse with
axes 4 and 2 and, on $z = 2$ an ellipse with axes $4\sqrt{2}$ and $2\sqrt{2}$.
4. Hyperboloid of two sheets. Trace on $x = 2$ point (2,0,0) and,
on $x = -4$ ellipse with axes $2\sqrt{3}$, $4\sqrt{3}$.
5. Cone. Trace on $x = -2$ an ellipse with axes 4 and 2,
on $x = 4$ an ellipse with axes 8 and 4.

7. Elliptic paraboloid. Trace on $x = -4$ an ellipse with axes 8 and 4, and no trace on $x = 4$.
9. Hyperboloid of two sheets. Trace on $x = 1$ nothing, on $x = 5$ a circle of radius 3. **21.** z-axis. **23.** Paraboloid.
25. Hyperboloid of one sheet. **27.** Sphere. **29.** Cone.

§86, page 225

1. $(1,-1,-4)$, $(-3,5,28)$, $(2,0,-2)$, $(-4,4,26)$.
2. (a) $(0,4,6)$. (c) $(2,0,0)$. (e) $(m,4-2m,6-3m)$.
13. (a) $(\frac{4}{3},\frac{26}{3},0)$. (b) $(\frac{1}{3},\frac{26}{3},1)$. **16.** (a) $x^2 + y^2 - 4x + 4z = 5$.
(b) $x^2 + y^2 - 5 + k(z - x) = 0$, for any value of k.
(c) $x^2 + y^2 - 5x + 5z = 5$. **17.** (c) ak.

§87, page 230

1. (a) $2\sqrt{3}$. (c) 9. **2.** (a) $\frac{1}{3}\sqrt{3}$, $\frac{1}{3}\sqrt{3}$, $\frac{1}{3}\sqrt{3}$. (c) $-\frac{1}{9}$, $\frac{4}{9}$, $-\frac{8}{9}$.
3. (a) $\sqrt{38} + \sqrt{19} + 9$. (b) 20. **6.** (a) 2. (c) 13. **7.** (a) $\frac{1}{9}$, $-\frac{4}{9}$, $\frac{8}{9}$.
8. (a) $\frac{1}{3}$, $\frac{2}{3}$, $-\frac{2}{3}$. (c) $\frac{1}{14}\sqrt{14}$, $\frac{1}{7}\sqrt{14}$, $\frac{3}{14}\sqrt{14}$. **9.** (a) 1, 0, 0.
(b) 0, 1, 0. (c) 0, 0, 1. **11.** x/r, y/r, z/r, where $r = \sqrt{x^2 + y^2 + z^2}$.
13. (a) $\pm\frac{1}{2}\sqrt{2}$. (b) ± 1. **15.** Perpendicular to: (a) x-axis.
(b) y-axis. (c) z-axis. **17.** (a) $\cos\gamma = 1$. (b) $\cos\beta = 1$.
(c) $\cos\alpha = 1$. **19.** $(x - 1)^2 + (y - 2)^2 + z^2 = 9$.
21. $x^2 + y^2 + z^2 - 3x + 2z = 0$.
22. $x/3 = y/1 = (z - 2)/(-2)$.

§88, page 234

1. 27.0°. **3.** 52.9°. **5.** 0°. **12.** $l = 2$, $m = -4$.
13. $\alpha = \beta = \gamma = 54.7°$. **14.** $z + 2y = 0$. **15.** (a) $(-4,-2,9)$.
(c) $(-12,4,-1)$. **16.** (a) $12/\sqrt{181}$, $-6/\sqrt{181}$, $-1/\sqrt{181}$.

§89, page 237

1. $\frac{2}{3}$, $-\frac{2}{3}$, $-\frac{1}{3}$; 4. **3.** $-\frac{3}{5}$, $-\frac{4}{5}$, 0; 5. **5.** $-\frac{3}{5}$, $\frac{4}{5}$, 0; 0.
7. 1, -3, 2. **9.** 22.2°. **11.** $2x + 3y - 6z = 49$.
13. (a) Intercepts a, b, c. (b) Through (x_1, y_1, z_1) and perpendicular to a line with direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$.
(c) A plane through (x_1, y_1, z_1) and perpendicular to a line with direction numbers A, B, C. **14.** $x + y + 2z = 2$. **15.** $x - y + z = 1$.
17. $7x + 6y - 3z = 0$. **19.** $x + 2y - 5z = 7$.

§91, page 241

1. Parallel: (a)(d), (b)(c).
Perpendicular: (a)(b), (a)(c), (b)(d), (b)(f), (c)(d), (c)(f), (e)(f).
3. 66.7°. **5.** 3. **7.** $x + y + 3z = 1$. **9.** $2x + y - z = 2$.
11. $x - y + z = 3$. **12.** System of planes parallel to $x + y + z = 0$.
13. Distant 5 from (0,0,0). **15.** Through (x_1,y_1,z_1). **16.** -6.
17. $-\frac{5}{3}$. **19.** 3. **20.** $8x + 3y - z = 27$, $4x - 9y + 5z = 3$. **21.** $\frac{55}{9}$.
22. (a) $3\sqrt{29}$. **23.** (a) $\frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$. **24.** 4.

§93, page 245*

1. $(0,3,3)$; $2, 4, -\frac{3}{2}$. **2.** $(3,5,-2)$; $2, -3, 6$. **3.** $(2,3,\frac{5}{3})$; $6, 3, 2$.
4. $(2,-3,0)$; $2, 5, 1$. **5.** $(5,4,0)$; $2, 2, 1$. **6.** (a) $(2,-3,0)$.
(b) $(0,-8,-1)$; $\frac{1}{2}(x - 2) = \frac{1}{5}(y + 3) = z$.
7. $(6,-17,17)$, $(23,-85,68)$; $\frac{1}{17}(x - 6) = -\frac{1}{68}(y + 17) = \frac{1}{51}(z - 17)$
9. $x - 2 = y - 3 = -z$. **11.** $x - 2 = \frac{1}{3}(y + 4) = \frac{1}{4}z$.
13. $x/2 = y/1 = z/3$. **14.** $\frac{1}{3}(x - 1) = \frac{1}{2}y = -\frac{1}{7}(z + 2)$.
15. $x = -y$, $z = 0$. **17.** $x = 2$, $\frac{1}{2}(y - 1) = \frac{1}{5}z$. **19.** $y = -1$, $z = 2$.
21. $z = -3$, $x - 1 = \frac{1}{2}(y - 2)$. **23.** $y = 4$, $x = z + 5$.
24. (a) $la + mb + nc = 0$. (b) $l/a = m/b = n/c$. **25.** 46.9°.
26. 6. **27.** $4x + y + 7z = 34$. **28.** $x - 2y + z + 2 = 0$.
29. $x - 5y + 16 = 0$. **31.** $4x + y + 7z = 34$.
33. $x - 2y + z + 2 = 0$.

§94, page 248

1. $y^2 + z^2 = x^4$. **2.** $x^2 + y^2 = z^2$. **3.** $y^2 + z^2 = 3x^2 + 4$.
5. $(x - 3)^2 + z^2 = 4y^2$. **7.** $x^2 + y^2 = 4z - 4$. **9.** $x = \pm y$.
11. $x^2 + y^2 = z^2 \tan^2 \gamma$. **13.** $19x^2 - 4xy + 16y^2 + 20z^2 + 4x + 8y = 4$.
14. $6x + 12y + 23z = 25$, $2x - 4y - 9z + 11 = 0$.

§95, page 250

1. (a) $(3\sqrt{2},45°,5)$. (c) $(4,-60°,6)$. (e) $(1,0°,0)$.
2. (a) $(2.5\sqrt{3},2.5,3)$. (c) $(0,5,-5)$. [The elements of the cylinders for Exercises 3 to 8 are all parallel to the z-axis, and the equations of the curves they intersect follow.] **3.** Circle $r = 1$, $z = 0$.
4. Circle $r = a \cos \theta$, $z = 0$. **5.** Line $r \sin \theta = 1$, $z = 0$.
6. Line $r \cos (\theta + 60°) = 1$, $z = 0$. **7.** Line $\theta = 30°$, $z = 0$.

* Evidently many answers are available for a number of the exercises of this section.

8. Line $\theta = 120°$, $z = 0$. **9.** $r^2 = 4z^2 + 2$.
11. $r(\cos\theta + \sin\theta) + z = 5$. **13.** $r^2 \sin 2\theta = 2z$.

§97, page 253

1. (a) $(\frac{3}{2},\frac{1}{2}\sqrt{3},1)$. (c) $(0,3\sqrt{3},-3)$. **2.** (a) $(3,0°,90°)$. (c) $(\sqrt{41},63.4°,20.4°)$. **3.** $\rho = a$. **5.** $\rho \sin\phi \sin\theta = 3$.
7. $\rho^2 \cos 2\phi = -a^2$. **9.** Sphere.
10. Vertical plane making an angle of 150° with the x-axis.
11. xy-plane. **13.** Sphere, radius $\frac{1}{2}a$. **14.** Cylinder $x^2 + y^2 = 1$.
15. A surface cut by planes $\theta = \alpha$ in circles with radii $\cos\alpha$ and centers at the origin. **16.** Sphere, radius $\frac{1}{2}a$.
17. (a) $(1,0,0)$. (c) $(1.155,1.155,1.155)$. **18.** $\rho = a$. **19.** $\gamma = 45°$.
20. $\rho(\cos\alpha + \cos\beta + \cos\gamma) = 5$. **21.** $\rho^2(1 - 2\cos^2\gamma) = a^2$.
22. $\rho^2(2\cos^2\alpha - 1) = a^2$. **23.** $\rho^2 = a^2 \sin^2\gamma$.

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